THE BASIC THEORY OF ELLIPTIC SURFACES

Notes of lectures by

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Preface

These notes are a faithful record of a course of lectures given in the Department of Mathematics at the University of Pisa in the fall of 1988. My aim was two-fold: first, to develop in some detail for a student of algebraic geometry the basic theory of elliptic surfaces over $\mathbb{C}$, and secondly, to present some recent results of joint work with U. Persson on configurations of singular fibers on elliptic surfaces.

I hope that the Lectures I - VII, with Lecture IX, serve the first purpose to a reasonable extent, and that Lectures VIII and X do justice to the second. In particular, I hope that anyone who has gone through these notes will be able to quickly get into a modern research article on elliptic surfaces.

Most of the lectures deal almost exclusively with Jacobian surfaces, that is, elliptic surfaces with a chosen section. This restriction was made mainly because there is so much to say, even about Jacobian surfaces, that I found myself constantly ignoring non-Jacobian questions in my haste to get to the description of the work with U. Persson. In addition, for this work the assumption of a section is harmless, so I felt that my audience was not being cheated too much.

I have included a bibliography at the end of the lectures, although I have made rather spare use of it in the body of the text. The reader should know that except for the material in Lectures VIII and X, the results are not my own. A basic set of references could be [BPV], [D], [K1], [Ka1], [MS], [S], and [T].

I would like to thank M. Cragnolini, M. Ojanguren, R. Pardini, M. Salvetti, and the students who attended these lectures for their kind attention and interest. I must express my gratitude to the C. N. R., for their grant support during my stay in Pisa. Finally it gives me great pleasure to thank Fabrizio Catanese for his kind invitation to visit Pisa and give these lectures. I has been a distinct honor to have been able to experience his friendship and his scholarship, both in the past and especially during my visit, and I look forward to extending our contact, personally and professionally, in the future.

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Lecture I: Definitions and examples.

I.1 Elliptic curves.

An elliptic curve $(E,\underline{0})$ over a field K is a complete curve E of genus one defined over K, together with a given point $\underline{0}$ defined over K. One is often sloppy and refers to the elliptic curve as E alone, suppressing the given point in the notation.

Note that we have called the chosen point $\underline{0}$; it is usually taken to be the origin of the group law on the K-rational points of the elliptic curve.

There are several quite common ways that elliptic curves arise in nature.

(I.1.1) Example: Fix a non-real complex number τ, and denote by $\Lambda(\tau)$ the subgroup of $\mathbb{C}$ generated by 1 and τ: $\Lambda(\tau) = \mathbb{Z} \oplus \mathbb{Z}\tau$. Then $E = \mathbb{C}/\Lambda(\tau)$ is an elliptic curve over $\mathbb{C}$. The chosen point is of course the class of 0.

(I.1.2) Example: Let K be any field, and let $E \subset \mathbb{P}^2_K$ be a smooth cubic curve with a given flex point p defined over K. Then (E,p) is an elliptic curve over K.

(I.1.3) Example: Let K be any field of characteristic unequal to 2, and let E be the double cover of $\mathbb{P}^1_K$ branched over ∞ and three other distinct points, which as a divisor of degree 3 on $\mathbb{P}^1_K$ is defined over K. Then $(E,\underline{0})$ is an elliptic curve over K, where $\underline{0}$ is the point over ∞. Such an elliptic curve can always be written as $y^2 = f(x)$, where f is a polynomial of degree 3 in K[x].

(I.1.4) Example: Consider the curve given by the equation $y^2 = x^3 + Ax + B$, where A and B are in a field K. This is the famous Weierstrass equation, and if $\Delta = 4A^3 + 27B^2$ is not zero in K, then this equation defines a smooth curve E, of genus one, with a single point $\underline{0}$ at ∞. $(E,\underline{0})$ is an elliptic curve over K. This equation is the one relating the Weierstrass $\wp$-function and its derivative (up to some constants), for an elliptic curve as in Example 1. In general, any elliptic curve over a field of characteristic unequal to 2 or 3 can be defined by a Weierstrass equation; one requires $\Delta \neq 0$ for E to be smooth.

I.2: The J-function.

Assume that K does not have characteristic 2 or 3, and let $(E,\underline{0})$ be defined by the Weierstrass equation $y^2 = x^3 + Ax + B$, with A and B in K. The basic theory of this equation says that the pair (A,B) is almost unique, i.e., it is almost determined by the elliptic curve E. In particular, two pairs (A_1,B_1) and (A_2,B_2) determine isomorphic elliptic curves if and only if there is a nonzero λ in K such that $A_2 = \lambda^4 A_1$ and $B_2 = \lambda^6 B_1$.

One direction of this is easy: if such a λ exists, then $y^2 = x^3 + A_2x + B_2 = x^3 + \lambda^4 A_1 x + \lambda^6 B_1 = \lambda^6((\lambda^{-2}x)^3 + A_1(\lambda^{-2}x) + B_1$, so replacing y by $\lambda^{-3}y$ and x by $\lambda^{-2}x$ we obtain $y^2 = x^3 + A_1x + B_1$. To see the other direction, we must essentially derive the Weierstrass equation from the elliptic curve $(E,\underline{0})$, which I will do very briefly.

Consider the vector spaces $V_n = H^0(E,\mathcal{O}_E(n\underline{0}))$; by Riemann-Roch, $\dim_{\mathbb{C}} V_n = n$ for $n \geq 1$. Let {1} be a generator for V_1, {1,f} generators for V_2, and {1,f,g} generators for V_3. Then the elements $\{1,f,g,f^2,fg,f^3\}$ forms a basis for V_6; since $g^2 \in V_6$ also, we have a relation, and we may assume this relation has the form $g^2 = f^3 + a_5fg + a_4f^2 + a_3g + a_2f + a_0$ for some a_i in K, by scaling g appropriately. By completing the square in g we may assume that a_5 and a_3 are zero, and then by completing the cube in f we may assume that a_4 is zero (here we use characteristic unequal to two or three). This gives the Weierstrass equation for $(E,\underline{0})$, with x being the new f and y the new g.

The function x is now determined (up to a constant which must be a square in K) by the requirement that a monic cubic in x with no x^2 term is the square of an element in V_3; the element y is determined after x is (up to $\pm$) by being that element of V_3. If we have $y^2 = x^3 + Ax + B$, and we replace x by λ^2x, then we must replace y by $\pm\lambda^3y$ to retain the form of the Weierstrass equation. This, as shown above, has the effect of replacing the pair of coefficients (A,B) with the pair $(\lambda^{-4}A,\lambda^{-6}B)$, and is the only amount of ambiguity in the pair.

Note that for a pair (A,B) then, there is just one invariant under this action of K^*, namely $\alpha = A^3/B^2$. It is more convenient to take as the invariant the quantity $4\alpha/(4\alpha+27) = 4A^3/\Delta$, because this linear fractional image of α is always in K, since we are assuming that E is smooth, hence $\Delta \neq 0$.

(I.2.1)<u>Definition:</u> $J(A,B) = 4A^3/(4A^3+27B^2)$.

We define J for an elliptic curve given by the Weierstrass equation $y^2 = x^3 + Ax + B$ to be $J(A,B)$. Note that this definition agrees with Kodaira's notation, but disagrees with all the number theorist's notation: they tend to have a factor of 12^3 in addition to this. In characteristics unequal to 2 and 3, this is not a real problem, but the reader should beware of the formulas using our J-function in any arithmetic situations.

The reader should note that $J(A,B) = 0$ if and only if $A = 0$ and $J(A,B) = 1$ if and only if $B = 0$.

Note that if two pairs (A_1,B_1) and (A_2,B_2) define isomorphic elliptic curves over K, then $J(A_1,B_1) = J(A_2,B_2)$. The converse is not quite true.

(I.2.2)<u>Lemma:</u> Assume that $J(A_1,B_1) = J(A_2,B_2)$. Then the elliptic curves $y^2 = x^3 + A_1x + B_1$ and $y^2 = x^3 + A_2x + B_2$ may not be isomorphic as elliptic curves over K; however if J is unequal to 0 or 1, they become isomorphic over at most a quadratic field extension of K.

<u>Proof:</u> It is easy to see that $J = J(A_1,B_1) = J(A_2,B_2)$ and $J \neq 0,1$ implies that there is a γ in K^* such that $A_2 = \gamma^2 A_1$ and $B_2 = \gamma^3 B_1$. Indeed, we have $(A_1/A_2)^3 = (B_1/B_2)^2$ and $\gamma = A_2B_1/A_1B_2$ works. If γ is a square in K, say $\gamma = \lambda^2$, then the curves are isomorphic over K. If not, they become isomorphic over $K(\sqrt{\gamma})$. ■

We will primarily be interested in the theory of elliptic surfaces over the field $\mathbb{C}$, but this inevitably leads to the discussion of elliptic curves over other fields (e.g., function fields of curves).

I.3: **Elliptic Surfaces.**

In this section we assume that the ground field is the complex numbers.

(I.3.1)<u>Definition:</u> An <u>elliptic</u> <u>surface</u> is a complex surface X together with a holomorphic map $\pi : X \longrightarrow C$ from X to a smooth curve C such that the general fiber of π is a smooth connected curve of genus one.

Note that we have not said that the general fiber of π is an elliptic curve, which might strike the reader as more logical. This would imply that there is given in each fiber a chosen point, which would mean that a section for the map π would be given. This is considered a special (though fundamental) case and we just agree to abuse the language in this way.

An elliptic surface $\pi:X \longrightarrow C$ is _smooth_ if X is a smooth surface. It is _relatively minimal_, or a _minimal elliptic_ surface if X is smooth and there are no (-1)-curves in the fibers of π.

We will call a curve on X _vertical for_ π, or simply _vertical_, if it lies in a fiber of π. We will otherwise call it _horizontal_. Thus a minimal elliptic surface has no vertical (-1)-curves. It may well have horizontal ones, however, and therefore not be a minimal surface in the sense of surface theory; it will only be minimal elliptic.

We say $\pi:X \longrightarrow C$ is an elliptic surface _with section S_, or simply _with section_, if a section $s:C \longrightarrow X$ of π is given; the image of s is the curve S on X.

Finally we say that X is an elliptic surface _over C_ if we wish to specify that C is the base curve. For example, an elliptic surface over $\mathbb{P}^1$ has a smooth rational base curve.

There is a local theory of elliptic surfaces, and a global theory, and of course the two are not unrelated. The local problems deal with the classification of the possible fibers of an elliptic surface $\pi:X \longrightarrow C$, the local monodromy (on $H^2(F,\mathbb{Z})$ for a fiber F) around a singular fiber, the local behavior of the J-map, etc. The global problems are ones familiar to any surface theorist: what are the numerical invariants of X, what is the canonical bundle formula, what is the global behavior of the J-map, the Kodaira dimension of X, etc. I will try in these lectures to deal in some detail with both the local and the global theory.

There are some obvious base change properties, which I leave to the reader to prove as an exercise.

(I.3.2)_Lemma:_ Let $\pi:X \longrightarrow C$ be an elliptic surface, and let $f:C_1 \longrightarrow C$ be a map of curves. Then the pull-back $\pi_1:X_1 = X \times_C C_1 \longrightarrow C_1$ is an elliptic surface. Moreover:

a) If π has a section S, then S induces a section S_1 of π_1.

b) If f is branched only over points p of C such that $\pi^{-1}(p)$ is smooth, and if X is smooth, then X_1 is smooth.

c) With the hypotheses of b), if X is smooth and minimal elliptic, then so is X_1.

I.4: The List of possible singular fibers.

Before beginning to address the issues mentioned above, I would like to give a series of examples to illustrate some of the features of the theory to the reader. It will be useful for the purposes of illustration and communication for the reader to know the possible singular fibers which can occur, and Kodaira's names for them. This I present below, without proof, simply so that I can speak of them intelligently in the examples to follow.

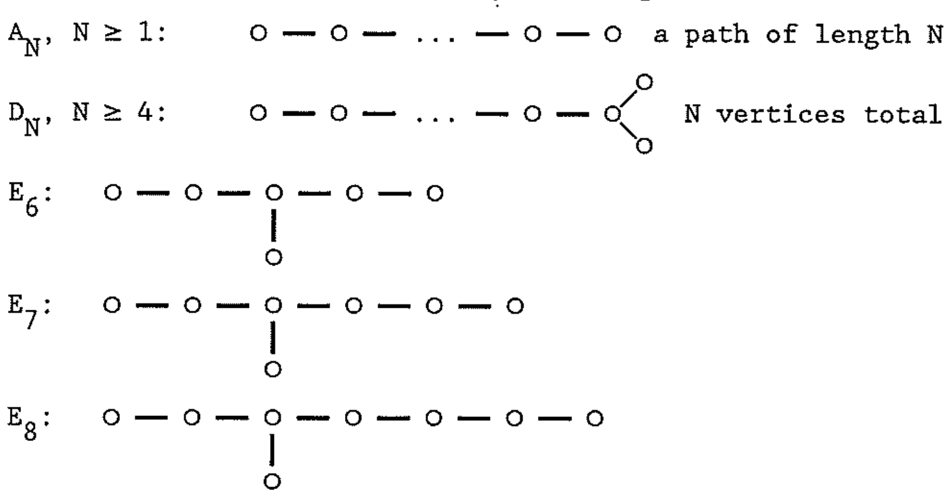
(I.4.1)<u>Table</u> of possible singular fibers of a smooth minimal elliptic surface. The names are those used by Kodaira.

Name	Fiber
I_0	smooth elliptic curve
I_1	nodal rational curve
I_2	two smooth rational curves meeting transversally at two points
I_3	three smooth rational curves meeting in a cycle; a triangle
$I_N, N\geq 3$	N smooth rational curves meeting in a cycle, i.e., meeting with dual graph $\tilde{A}_N$
$I_N^*, N\geq 0$	N+5 smooth rational curves meeting with dual graph $\tilde{D}_{N+4}$
II	a cuspidal rational curve
III	two smooth rational curves meeting at one point to order 2
IV	three smooth rational curves all meeting at one point
IV^*	7 smooth rational curves meeting with dual graph $\tilde{E}_6$
III^*	8 smooth rational curves meeting with dual graph $\tilde{E}_7$
II^*	9 smooth rational curves meeting with dual graph $\tilde{E}_8$
${}_M I_N, N\geq 0$	topologically an I_N, but each curve has multiplicity N

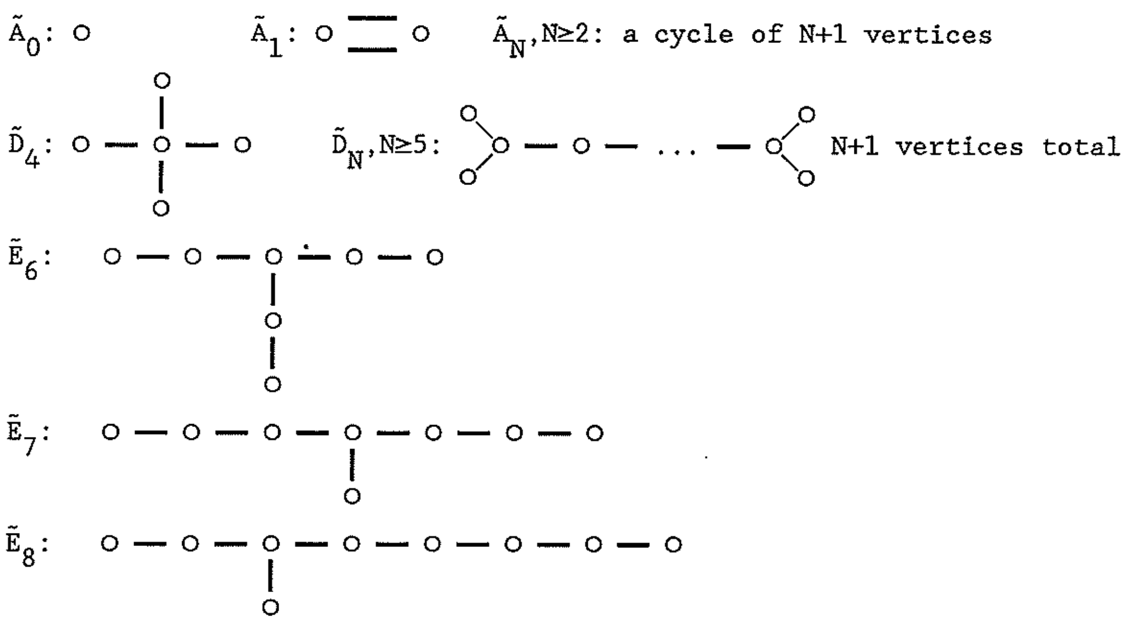
All components of reducible fibers have self-intersection -2; the irreducible fibers have self-intersection 0, of course.

The dual graphs referred to above are those of the extended Dynkin diagrams. For ease of reference I'll give below tables of the Dynkin diagrams and the extended Dynkin diagrams.

(I.4.2)<u>Table</u> The Dynkin Diagrams



(I.4.3)<u>Table</u> The Extended Dynkin Diagrams (with multiplicities)



The multiplicities of the vertices of the extended Dynkin diagrams are given below.

$\tilde{A}_N$, $N \geq 0$: coefficient 1 on each vertex.

$\tilde{D}_N$, $N \geq 4$: coefficient 1 on each vertex of degree 1, 2 on all others

$\tilde{E}_6$: $\begin{matrix}1&2&3&2&1\\&&2&&\\&&1&&\end{matrix}$; $\tilde{E}_7$: $\begin{matrix}1&2&3&4&3&2&1\\&&&2&&&\end{matrix}$, $\tilde{E}_8$: $\begin{matrix}2&4&6&5&4&3&2&1\\&&3&&&&&\end{matrix}$

A fiber of an elliptic surface $\pi: X \longrightarrow C$ is more than just a union of curves, of course, it is a divisor on X, and as such each component of the fiber has a multiplicity. This information is missing in the above table (I.4.1). For types I_0, I_1, I_2, I_3, II, III, and IV, the multiplicity of every component is 1. For the types ${}_M I_N$, the multiplicity of every component is M. For all other types, the multiplicities are the multiplicities indicated at the end of Table (I.4.3) for the appropriate extended Dynkin diagram.

I.5: **Examples.**

In this section I will present several examples to illustrate the existence of singular fibers of many of the types indicated above in section 4, and also several other more global facts concerning elliptic surfaces. Most of the examples come from pencils of plane cubic curves.

(I.5.1)<u>Example:</u> A pencil of plane cubics.

Let C_1 be a smooth cubic curve in $\mathbb{P}^2$, and let C_2 be any other cubic. The pencil generated by C_1 and C_2 has 9 base points (some possibly infinitely near) and after blowing them up the fundamental locus of the rational map from $\mathbb{P}^2$ to $\mathbb{P}^1$ given by the pencil is resolved, and we obtain a morphism $\pi: X \longrightarrow \mathbb{P}^1$ which exhibits the rational surface X as being elliptic over $\mathbb{P}^1$. It is minimal elliptic, since there are no vertical (-1)-curves; if there were, we would have blown up too much, but we blew up exactly nine times, which is required since $C_1 \cdot C_2 = 9$. The canonical class of X is $-C_1$; in particular, $K_X^2 = 0$.

By varying C_2 appropriately, we may see many of the possible singular fibers in the Table (I.4.1). Indeed, if we take C_2 to be reduced, meeting C_1 in 9 distinct points transversally, then then fiber of π corresponding to C_2 will be simply the proper transform of C_2 itself. Hence if C_2 is a nodal cubic, we obtain a fiber of type I_1, and if C_2 is a cuspidal cubic, we obtain a fiber of type II. If C_2 is a conic plus a line, we obtain the types I_2 (if the line is not tangent to the conic) or type III (if the line is tangent). If C_2 consists of three distinct lines, then we obtain type I_3 (if the lines are not concurrent) or type IV (if they are).

Note that since $K_X = -C_1$, we have $K_X \cdot A = 0$ for any component of any fiber. Hence we verify that for all components of reducible fibers in the above cases (which are all smooth and rational), we have self-intersection -2.

How many singular fibers does such pencil have? As in every enumerative question in algebraic geometry, the answer is fairly nice if you count properly, and the problem is to determine what "properly" means. First let me present a criterion for a general cubic curve in the plane to be singular.

Let x_1, x_2, x_3 be the homogeneous coordinates in $\mathbb{P}^2$, and consider a homogeneous cubic equation $F(\underline{x}) = 0$. Form the Hessian $H_F = \det(\partial^2 F/\partial x_i \partial x_j)$, which is also a cubic, and the nine further cubics $G_{ij} = x_i \partial F/\partial x_j$. Let D_F be the determinant of the 10×10 matrix of coefficients of H and the G_{ij}; D_F is a function of the coefficients of F, and since H is cubic and the G_{ij} linear in these coefficients, D_F is homogeneous of degree 12 in the coefficients of F. It is well-defined up to sign.

(I.5.2)<u>Lemma:</u> The cubic curve C defined by $F = 0$ is singular if and only if $D_F = 0$.

Proof: C is singular if and only if the three partials of F have a common solution in $\mathbb{P}^2$; if they do, then so do H and all the G_{ij}, so that these 10 cubics share that common point. Hence they cannot be linearly independent, so D_F must be zero.

The above argument shows that the discriminant locus in $\mathbb{P}^9$ for cubic curves is contained inside the zero locus of D; we wish to show that they are equal. A calculation of the above determinant for $F = y^2 - x^3 - Ax - B$ gives that D_F is a constant (I believe it is $3^3 2^7$, but don't hold me to it) times $4A^3+27B^2$, which is the correct discriminant Δ for the Weierstrass equation. Since every smooth cubic can be written in Weierstrass form, this shows that if a cubic $F = 0$ is smooth, then $D_F \neq 0$. ∎

This shows that the discriminant locus for plane cubics has degree 12, and so we expect that in a general pencil of smooth cubics we will find 12 singular members; in a non-general pencil, we expect still 12 singular members, counted properly. Indeed, since the most general singular cubic is a nodal cubic, one expects in a general pencil to find 12 nodal cubics, contributing 12 singular fibers to the associated elliptic surface, each of type I_1.

(I.5.3)<u>Example:</u> A non-reduced fiber.

Let C_2 now be 2L+M, for L and M distinct lines in $\mathbb{P}^2$, and let C_1 be a general smooth cubic meeting L and M in three points transversally each. After resolving the base points of the pencil generated by C_1 and C_2, one will find as the singular fiber corresponding to C_2 a fiber of type I_0^*. This fiber type is not reduced; the proper transform of L occurs to multiplicity 2 in the fiber. The other four curves of the fiber are M and the first three exceptional curves lying over the three points of $C_1 \cap L$.

(I.5.4)<u>Example:</u> A family with constant J.

Let C_1 be defined by $y^2z = 0$ and C_2 by $x(x^2+axz+z^2) = 0$, for a fixed constant $a \neq \pm 2$. The pencil is then given by $F = \lambda y^2z + \mu x(x^2+axz+z^2)$. The calculation of D_F in this case gives D_F equal to a constant times $\lambda^6\mu^6$, showing that there are only two singular fibers, corresponding to the elements C_1 and C_2 of the pencil. Those singular fibers, the reader will find after resolving the base points of the pencil, are both of type I_0^*; each is counting for 6 of the 12 singular fibers.

Note that the J-invariant of the general fiber does not vary with (λ,μ)! Hence this is a family of <u>isomorphic</u> elliptic curves, but which degenerates non-trivially. The surface is not a product, since there are degenerations.

(I.5.5)<u>Example:</u> Higher Polygons.

Let C_2 be the triangle $xyz = 0$, and let C_1 be a smooth cubic. If C_1 goes through a vertex of the triangle, then the exceptional curve over the vertex will contribute a component to the fiber of the elliptic surface corresponding to C_2; if in addition C_1 is tangent to one of the lines of C_2 at the vertex, then two (of the three) exceptional curves lying over that vertex occur in the fiber corresponding to C_2. In every case, the triangle of C_2 is expanded into a higher polygon, giving a fiber of type I_N for some $N \geq 3$. We can arrange in this way to have a fiber of type I_N with $3 \leq N \leq 9$, 9 being obtained when C_1 is tangent to each of the three lines of C_2 at each of the vertices in turn. For example, if C_1 is given by the cubic $xy^2+yz^2+zx^2 = 0$, we obtain an I_9 fiber.

(I.5.6)<u>Example:</u> Base change of an I_N fiber.

Let $\pi:X \longrightarrow D$ be a smooth minimal elliptic surface over a disc with a singular fiber of type I_N over 0. If t is a coordinate on the disc D centered at 0, make the base change $t = s^2$. This amounts to taking the double cover of X branched over the N components of the singular fiber, and after resolving the N ordinary double points (occurring over the nodes of X_0) one easily sees that the singular fiber after the base change is of type I_{2N}.

In this way one sees that one can obtain singular fibers of type I_M with arbitrarily high M.

In fact, as we will see later, if one base changes a fiber of type I_N to order M (replacing t by s^M), one obtains a fiber of type I_{MN}.

(I.5.7)Example: Double base change of Example(I.5.4).

Let $f:\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the double cover branched over 0 and ∞, and base change the elliptic surface X of Example (I.5.4) via f. We are ramifying over exactly the two singular fibers of f, and they are both of type I_0^*. After normalizing the double cover, we are taking the double cover of X branched along the 8 multiplicity one curves in the two fibers. Locally, this double cover is a smooth elliptic curve (lying over the multiplicity two component) with self-intersection -4 and 4 (-1)-curves (lying over the four multiplicity one components). After blowing down the four (-1)-curves in the two fibers, we obtain an elliptic surface over $\mathbb{P}^1$, with constant J, and with no singular fibers. This surface is a product of $\mathbb{P}^1$ with an elliptic curve, as is rather easy to verify using the classification of surfaces.

(I.5.8)Example: A pencil with an I_1^* fiber.

We have seen, in Example (I.5.3), that if we take C_2 to be 2L+M, where L and M are distinct lines, and C_1 to meet L transversally in 3 distinct points, we obtain a fiber of type I_0^* corresponding to C_2. If instead we take C_1 tangent to L, we obtain a fiber of type I_1^*.

(I.5.9)Example: IV^*, III^*, and II^* fibers.

Let C_2 be a triple line 3L. IF we take C_1 to meet C_2 in three distinct points, we obtain a fiber of type IV^* corresponding to C_2 in the elliptic surface. If C_1 is tangent of C_2, and meets C_2 at one other point, then we obtain a fiber of type III^*. If C_1 has L as a flex line, then we obtain a fiber of type II^*.

(I.5.10)Example: A multiple fiber.

Let C be a smooth cubic, and choose 9 points $p_1,\dots,p_9$ on C so that $\sum p_i$ is not a divisor in 3H, but $2\sum p_i$ is a divisor in 6H (here H is the hyperplane divisor of C). Consider the set of sextic curves in the plane which are double at the 9 points p_i. There are 27 parameters for sextics, and imposing a double point is three parameters, so there is at least one such sextic. Of course 2C is one.

I claim in fact there are two, hence a pencil of such. There is surely a pencil of sextics double at $p_1,\dots,p_8$, passing through p_9, and having some different tangent at p_9 other than that of C. (This is only 26 conditions.) However, by Abel's theorem on C, any such sextic must meet C twice at p_9, hence must in fact be double there also.

Let S be such a sextic, and consider the pencil generated by S and 2C. A smooth sextic has arithmetic genus 10, so that the general member of this pencil has geometric genus 1, and after resolving the base points of the pencil (by blowing up $p_1,\dots,p_9$) we obtain an elliptic surface with a multiple smooth fiber, of type ${}_2I_0$; this is the proper transform of 2C.

This example can be generalized to obtain fibers of type ${}_MI_0$ for any M, by considering a pencil of curves of degree 3M which have 9 M-fold points as base locus, cutting out a divisor D o\n a smooth cubic such that MD is 3M times the hyperplane divisor.

(I.5.11)<u>Example:</u> A multiple triangle.

Let Q be a smooth conic in the plane, and let L_1, L_2, and L_3 be three distinct tangent lines to Q. Form the pencil of sextics generated by 3Q and $2(L_1+L_2+L_3)$. This gives an elliptic surface with two singular fibers, one of type IV^* (over Q) and a multiple fiber of type ${}_2I_3$ (the proper transforms of the L_i).

I.6: The Classification of the Fibers

In this section I will present a proof of Kodaira's classification of the singular fibers, which was written down in Table (I.4.1). We first give a lemma which is quite general for a fibration whose fibers have arbitrary genus. Assume X_0 is the singular fiber, and write $X_0 = \sum n_i C_i$, with $n_i > 0$ and C_i irreducible for each i. The intersection form on X induces a symmetric bilinear form $\langle -,- \rangle$ on the $\mathbb{Q}$-vector space V with basis the set of components $\{C_i\}$.

(I.6.1)<u>Lemma:</u>

a) The form $\langle -,- \rangle$ is negative semi-definite on V.

b) The kernel of $\langle -,- \rangle$ has dimension one, and is spanned by X_0.

Proof: We'll assume that X_0 is a fiber of a fibration of curves on an algebraic surface, so that we have access to the Hodge index theorem. One form of this theorem is that if D_1 and D_2 are two $\mathbb{Q}$-divisors on the surface X, with $D_1^2 > 0$ and $D_1 \cdot D_2 = 0$, then $D_2^2 \le 0$; moreover $D_2^2 = 0$ if and only if D_2 is homologous to zero on X.

Now assume that there is a class D_1 in V with $D_1^2 > 0$, and apply the Hodge index theorem with $D_2 = X_0$. Since $X_0 \cdot C_i = X_t \cdot C_i = 0$ for each i, X_0 is certainly in the kernel of <-,->. Hence D_1 and D_2 satisfy the hypotheses of the Hodge index theorem, and we conclude that $D_2 = X_0$ is homologous to zero on X, which is absurd since it is a positive divisor.

This proves that the form <-,-> is negative semi-definite, and we must now show that we have only a rank one kernel. In fact I claim that if D is any class in V with $D^2 = 0$, then D is in the span of X_0. Assume on the contrary that D is not a multiple of X_0. Then there is any $\alpha \in \mathbb{Q}$ such that $G_\alpha = D + \alpha X_0$ can be written as $\sum r_i C_i$, with $r_i \neq 0$ for every i, and such that there is an i with $r_i > 0$ and a j with $r_j < 0$. Write $G = P - N$, where P and N are both positive combinations of the C_i's, with disjoint support; by our assumption on α, neither P nor N is zero. Note that since X_0 is connected, $P \cdot N > 0$. Then, since $D^2 = 0$ and X_0 is in the kernel of the form, we have $G_\alpha^2 = 0$. But $G_\alpha^2 = (P-N)^2 = P^2 - 2PN + N^2 < 0$ by part a) and the inequality $P \cdot N > 0$. This contradiction proves that D must be in the span of X_0. ∎

(I.6.2)<u>Remark:</u> One can avoid the use of the Hodge index theorem here, if one is willing to do some more work with quadratic forms. The lemma is true quite generally for any singular fibers of a fibration of curves on a smooth surface, e.g., for fibrations over the disc.

Now to some elementary graph theory. Let G be a graph, possibly with loops and/or with multiple edges. Form the $\mathbb{Q}$-vector space V_G with basis the vertices of G, and define a symmetric bilinear form on V_G by declaring

$$v^2 = -2 + 2(\text{\# of loops at } v) \text{ for all vertices } v,$$

and $$vw = \text{the \# of edges joining } v \text{ to } w \text{ in } G, \text{ if } v \neq w.$$

This space V_G, together with this form, is called the <u>associated</u> <u>form</u> to G.

The reader should check, if he or she has never done so before, that the graphs of Table (I.4.2), called the Dynkin diagrams, or the A-D-E graphs, have a negative definite associated form.

Note that the element of V_G defined by attaching to each vertex its multiplicity, according to Table (I.4.3), is in the kernel of the form, for each extended Dynkin diagram. Thus, since each extended Dynkin diagram contains an ordinary one with one fewer vertex, we see that the associated form to each extended Dynkin diagram is negative semi-definite, with a dimension one kernel, spanned by the vector of multiplicities.

Note the following amusing fact:

(I.6.3)Lemma:

a) Every connected graph either is contained in or contains an extended Dynkin diagram.

b) Every connected graph without loops or multiple edges either is contained in or contains an extended Dynkin diagram without loops or multiple edges (i.e., not $\tilde{A}_0$ or $\tilde{A}_1$).

Proof: Clearly b) implies a), since if a graph has a loop it contains $\tilde{A}_0$. If a graph has a multiple edge, it contains $\tilde{A}_1$, so we may assume that G does not contain any loops or multiple edges. If G contains a cycle, then it contains $\tilde{A}_N$ for some N, so we may assume that G is a tree. If G contains a vertex of degree 4 or more, then it contains $\tilde{D}_4$, so we may further assume that all vertices of G have degree 1, 2, or 3.

If G has two or more vertices of degree 3, it contains $\tilde{D}_N$ for some N; if all vertices of G are of degree 1 or 2, then G is a path, and so is contained in $\tilde{A}_N$ for some N. Hence we may assume that G has exactly one vertex of degree 3. In this case G is a $T_{p,q,r}$ graph, i.e., a graph with one central vertex v of degree 3, and 3 "arms" emanating from v, of lengths p, q, and r (counting v in each arm; we have then a total of p+q+r-2 vertices). Note that $\tilde{E}_6$ is $T_{3,3,3}$, $\tilde{E}_7$ is $T_{2,4,4}$, and $\tilde{E}_8$ is $T_{2,3,6}$.

We order p, q and r so that $2 \le p \le q \le r$. If $p \ge 3$, then G contains $\tilde{E}_6$; hence we may assume that $p = 2$. If $q \ge 4$, then G contains $\tilde{E}_7$; if $q = 2$, then G is contained in $\tilde{D}_N$ for some N. Hence we may assume that $q = 3$.

If $r \le 4$, then G is contained in $\tilde{E}_7$, while if $r \ge 5$, G contains $\tilde{E}_8$. This completes the analysis. ■

The above lemma implies that the extended Dynkin diagrams are the only graphs whose associated form is negative semidefinite with dimension one kernel.

(I.6.4)Corollary: Let G be a connected graph whose associated form is negative semidefinite, with kernel of dimension one. Then G is an extended Dynkin diagram.

Proof: If G has a loop, then since G is connected and the associated form is negative semidefinite, G must have simply one vertex and one loop, i.e., G must be $\tilde{A}_0$. We may therefore assume that G has no loops. If G has a multiple edge, joining say vertices v and w, then the negative semidefiniteness (applied to the element v+w of V_G) shows that there must be exactly two edges only, and no other vertices; hence G is $\tilde{A}_1$. We may therefore assume that G has no multiple edges.

In this case, if G contains a Dynkin diagram, then the square zero class x_0 of the Dynkin diagram must be the generator for the kernel of V_G, and so G can have no vertices other than those of the Dynkin diagram. Hence G is the Dynkin diagram, since it has no multiple edges.

If G is contained in a Dynkin diagram, the generator x of the kernel of V_G must be a multiple of the square zero class of the Dynkin diagram; since this class has a strictly positive coefficient on all the vertices of the Dynkin diagram, G must contain all those vertices, and so G must again equal the Dynkin diagram.

These exhaust all the cases, by the previous lemma. ∎

Let now $\pi:X \longrightarrow C$ be a smooth minimal elliptic surface, with a special fiber over $0 \in C$. Write the fiber as $X_0 = \sum n_i C_i$, with $n_i > 0$ and C_i irreducible for each i. Let M be the g.c.d. of the multiplicities n_i; M is the <u>multiplicity</u> of the fiber, and we may write $X_0 = MF$, where $F = \sum r_i C_i$, with the r_i's having no common factor. The fiber is called a <u>multiple</u> fiber if $M > 1$.

We have one more fact which is required; its proof can be found in [BPV, III.8.3].

(I.6.5)<u>Lemma:</u> If X_0 is a fiber of multiplicity M, then $\mathcal{O}_F(F)$ is a torsion line bundle with order M in Pic(F). In particular, if F is simply connected (so that Pic(F) has no torsion), then $M = 1$.

We are now in a position to verify the table presented earlier.

(I.6.6)<u>Theorem:</u> The only possible fibers for a smooth minimal elliptic surface are those listed in Table (I.4.1).

Proof: Again write a fiber X_0 as MF, and write F as $\sum r_i C_i$. If F is irreducible, then, since the arithmetic genus of F is one, it is either a smooth elliptic curve (type I_0 if $M = 1$), a nodal rational curve (type I_1 if $M = 1$), or a cuspidal rational curve (type II if $M = 1$). So assume that F is reducible. By the adjunction formula, $K_X \cdot X_\eta = 0$ for a general fiber X_η; hence $K_X \cdot F = 0$ also, so that $0 = \sum r_i C_i \cdot K_X = \sum r_i (2p_a(C_i) - 2 - C_i^2)$.

I claim that all the integers $2p_a(C_i)-2-C_i^2$ are non-negative. Indeed, if $2p_a(C_i)-2-C_i^2 < 0$, then $-2 \le 2p_a(C_i)-2 < C_i^2 \le -1$, (since the form is negative semidefinite and F is reducible; here we use (I.6.1)). Therefore we must have $p_a(C_i) = 0$, so C_i is a smooth rational curve, and $C_i^2 = -1$; however this is ruled out by minimality.

Since every r_i is strictly positive, we must therefore have $2p_a(C_i)-2-C_i^2 = 0$ for each i, forcing each C_i to be a smooth rational curve with self-intersection -2.

Form the dual graph G to the fiber F, that is, a vertex v_i for every component C_i, and $C_i \cdot C_j$ edges joining v_i to v_j. By Lemma (I.6.1) and Corollary (I.6.4), G must be one of the extended Dynkin diagrams, (not $\tilde{A}_0$). If G is $\tilde{A}_1$, then F must be either I_2 (if the two components meet at two points) or III (if the two components meet at one point). If G is $\tilde{A}_2$, then F is either I_3 (if the components meet in a cycle) or IV (if they all meet at one point). In all other cases there is no ambiguity to how the components meet, and we obtain the types I_N, $N \ge 4$, (from the diagrams $\tilde{A}_{N-1}$), I_N^*, (from the diagrams $\tilde{D}_{N+4}$), and the types IV^*, III^* and II^* (from the diagrams $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$).

This completes the analysis in case $M = 1$. If $M \ne 1$, then F must not be simply connected, so that only $F = I_N$, $N \ge 0$, are allowed. This gives the types ${}_MI_N$, and completes the proof. ■

Lecture II: The Weierstrass equation.

II.1: Uniqueness of minimal models.

Let $\pi:X \longrightarrow C$ be a smooth elliptic surface. If X is not minimal elliptic, then there must be some vertical (-1)-curves; a smooth minimal elliptic surface may be obtained by blowing down any vertical (-1)-curves one finds (there can be only finitely many such in every fiber of π, of course).

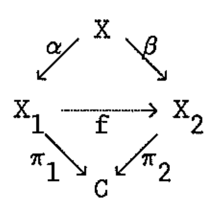
I claim that this smooth minimal elliptic surface is unique up to isomorphism.

(II.1.1)Definition: Let $\pi_1:X_1 \longrightarrow C$ and $\pi_2:X_2 \longrightarrow C$ be two elliptic surfaces over C. We say they are birational as elliptic surfaces over C if there is a birational map $f:X_1 \dashrightarrow X_2$ with $\pi_1 = \pi_2 \circ f$.

In fact we have the following.

(II.1.2)Proposition: Assume $\pi_1:X_1 \longrightarrow C$ and $\pi_2:X_2 \longrightarrow C$ are smooth minimal elliptic surfaces, which are birational as elliptic surfaces over C; let f be a birational map between them. Then f is an isomorphism.

Proof: The hypothesis that f preserves the elliptic structure implies that the map f can be resolved by only blowing up points and blowing down curves in the fibers of the π_i's. Let X be a surface dominating both X_1 and X_2, such that the number of blow-ups necessary to go from X_1 to X is minimal. We have the following diagram:



Here α and β are the blowups of X_1 and X_2, respectively. We wish to show that β is an isomorphism. If not, factor β into a sequence of blow-ups, and let E be the first exceptional curve blown down by β. If E is also exceptional for α, then after blowing down E we obtain a surface X' which is "closer" to X_1, but still dominates both X_1 and X_2. This violates the minimality of X, and so E is not exceptional for α.

Therefore E must be the proper transform of a component of a fiber of π_1. However, consider the possible components, as listed in Table (I.4.1): the image $\overline{E}$ of E in X_1 is either a smooth rational curve with self-intersection -2, a nodal or cuspidal rational curve with self-intersection 0, or a smooth elliptic curve. Since E is smooth and rational, this last case is not possible, and since $E^2 = -1$ on X, the first case is not possible either: the self-intersection can only go down upon blowing up. Therefore $\overline{E}$ is a rational curve with a single double point, with self-intersection 0. To obtain the smooth proper transform E, at some stage in the sequence of blow-ups of α the double point of $\overline{E}$ must be blown up; when this happens, the self-intersection will drop by 4, and so obtaining $E^2 = -1$ is not possible in this case either. Therefore E cannot exist, and β must be an isomorphism.

By symmetry, α is an isomorphism also, and hence so is f. ■

(II.1.3)<u>Corollary:</u> Given an elliptic surface $\pi:X \longrightarrow C$, there is a unique smooth minimal elliptic surface $\pi_1:X_1 \longrightarrow C$ birational to $\pi:X \longrightarrow C$ as elliptic surfaces over C.

If $\pi:X \longrightarrow C$ is an elliptic surface over C, then the generic fiber X_η of π is a curve of genus one over the function field K(C) of C. Conversely, if one has a curve of genus one over the function field K(C), then the above Corollary shows that there is a unique smooth minimal elliptic surface over C with that curve as generic fiber. Therefore we have a 1-1 correspondence:

$$\text{(II.1.4)} \qquad \left\{\begin{matrix}\text{smooth minimal}\\ \text{elliptic surfaces}\\ \text{over } C\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{curves of}\\ \text{genus one}\\ \text{over } K(C)\end{matrix}\right\}$$

(both sets taken up to isomorphism)

II.2: The Weierstrass equation for an elliptic curve over a field

In this section I want to present the theory of the Weierstrass equation for an elliptic curve over a field $\mathbb{K}$ of characteristic unequal to 2 or 3. This is a very well-known story, but I include it here for convenience.

Let E be a reduced irreducible complete curve of arithmetic genus one over $\mathbb{K}$, and let p be a smooth closed point of E. This therefore includes the case when E is a nodal rational or cuspidal rational curve. For every $n \ge 0$, let $V_n = H^0(E,\mathcal{O}_E(n\cdot p))$; by Riemann-Roch, we have

(II.2.1): $\qquad V_0 \cong V_1 \cong \mathbb{K}$ and $\dim_{\mathbb{K}} V_n = n$ for all $n \ge 1$.

Indeed, we interpret V_0, and therefore V_1, as the constant functions on E, and we consider V_n as the space of rational functions on E with only a single pole of order at most n at p. In this way we have naturally $V_i \subseteq V_{i+1}$ for every $i \geq 0$. Moreover, multiplication gives us a natural map $V_i \otimes V_j \longrightarrow V_{i+j}$, and from $\mathrm{Symm}^k V_i$ to V_{ki}.

(II.2.2)<u>Lemma:</u>

(a) There is a nonzero element $y \in V_3 - V_2$ such that y^2 (considered to be in V_6) is in the image of $\mathrm{Symm}^3 V_2$.

(b) There is a nonzero element $x \in V_2 - V_1$ such that a monic cubic in x with no x^2 term is equal to y^2.

(c) If (x_1,y_1) and (x_2,y_2) are two pairs of elements satisfying (a) and (b) above, then there is a nonzero element λ in $\mathbb{K}$ such that $x_2 = \lambda^2 x_1$ and $y_2 = \lambda^3 y_1$.

Proof: The set $\{1\}$ is a basis for V_0 and V_1. Let $f \in V_2 - V_1$ and $g \in V_3 - V_2$. Note that $\{1,f\}$ is a basis for V_2 and $\{1,f,g\}$ is a basis for V_3. Then $f^2 \in V_4 - V_3$, $fg \in V_5 - V_4$, and $f^3 \in V_6 - V_5$, so that $\{1,f,g,f^2,fg,f^3\}$ is a basis for V_6. Since $g^2 \in V_6$, we must have constants a_i in $\mathbb{K}$ such that $g^2 = a_6 f^3 + a_5 fg + a_4 f^2 + a_3 g + a_2 f + a_1$. Note that $a_6 \neq 0$, since g^2 is not in V_5.

By replacing f by $a_6 f$ and g by $a_6^2 g$, we may assume that $a_6 = 1$. Let $y = g - (a_5 f + a_3)/2$; this completes the square in g and we have $y^2 = f^3 + b_2 f^2 + b_1 f + b_0$ for some constants b_i in $\mathbb{K}$. This proves (a).

Let $x = f + b_2/3$; this completes the cube in f and we have $y^2 = x^3 + Ax + B$ for some constants A,B in $\mathbb{K}$. This proves (b).

Note that $\{1,x\}$ is a basis for V_2 and $\{1,x,y\}$ is a basis for V_3. Let us now address the uniqueness statement (c). Note that the relationship of (c) gives an action of $\mathbb{K}^*$ on the possible pairs (x,y): $\lambda\cdot(x,y) = (\lambda^2 x, \lambda^3 y)$. What we are claiming is that there is just one orbit for this action. Therefore it suffices to show that every possible pair is in the orbit of the one (x,y) which we found above.

Suppose (x_1,y_1) is another such pair. We may write $y_1 = \alpha y + \beta x + \gamma$, with $\alpha \neq 0$. Then

$$y_1^2 = \alpha^2 y^2 + 2(\beta x+\gamma)y + (\beta x+\gamma)^2 = \alpha^2(x^3+Ax+B) + 2(\beta x+\gamma)y + (\beta x+\gamma)^2,$$

and this is in $\mathrm{Symm}^3 V_2$ (i.e., it is a polynomial of degree 3 in x) if and only if $\beta = \gamma = 0$. Hence y_1 is a multiple of y: $y_1 = \alpha y$.

Now consider x_1; we may write $x_1 = ax + b$, with $a \neq 0$. By assumption we have $y_1^2 = x_1^3 + A_1x_1 + B_1$ for some A_1, B_1 in $\mathbb{K}$. After substituting we find that $\alpha^2y^2 = (ax+b)^3 + A_1(ax+b) + B_1 = a^3x^3 + 3a^2bx^2 + \ldots$, which is supposed to be $\alpha^2(x^3+Ax+B)$. In particular, we must have $b = 0$ to have no x^2 term; therefore x_1 is a multiple of x: $x_1 = ax$.

In addition, the equation above shows that $\alpha^2 = a^3$, so that there exists a nonzero λ in $\mathbb{K}$ such that $\alpha = \lambda^3$ and $a = \lambda^2$. This proves (c). ∎

(II.2.3)<u>Corollary:</u> There are elements A and B in $\mathbb{K}$ such that E is defined by the equation $y^2 = x^3 + Ax + B$. The pair (A,B) is unique up the action of $\mathbb{K}^*$ defined by $\lambda\cdot(A,B) = (\lambda^4A,\lambda^6B)$.

Proof: The affine curve E-p can be written as Spec(R), where R is the ring $R = \bigcup_{n=0}^{\infty} V_n$, since the point p, considered as a divisor on E, is ample. Moreover it is clear, by (II.2.1), that the elements $(1,x,x^2,\ldots,x^m,y,xy,x^2y,\ldots,x^{m-1}y)$ form a basis for V_{2m+1}, and the elements $(1,x,x^2,\ldots,x^m,y,xy,x^2y,\ldots,x^{m-2}y)$ form a basis for V_{2m}. Hence x and y generate R, and the since the relation above gives the correct Hilbert function for R (eliminating the need for y^2 at every stage), it must be the only relation between x and y. Hence $R = \mathbb{K}[x,y]/(y^2-x^3-Ax-B)$ as claimed.

To verify the uniqueness statement, use the uniqueness of x and y. Suppose E-p was defined by $y_1^2 = x_1^3 + A_1x_1 + B_1$. Then (x_1,y_1) would satisfy the conditions of the previous lemma, so that there would exist λ in $\mathbb{K}^*$ such that $x_1 = \lambda^2x$ and $y_1 = \lambda^3y$. Then

$$x_1^3 + A_1x_1 + B_1 = y_1^2 = \lambda^6y^2 = \lambda^6(x^3 + Ax + B) = x_1^3 + \lambda^4Ax_1 + \lambda^6B,$$

showing that $A_1 = \lambda^4A$ and $B_1 = \lambda^6B$. ∎

(II.2.4)<u>Definition:</u> We call the pair (x,y) with the properties of Lemma (II.2.2) a <u>Weierstrass</u> <u>basis</u> for (E,p) (or simply E).

It is instructive (and amusing) to verify the previous calculations in the case of a nodal or cuspidal rational curve. Assume that E is a nodal rational curve, obtained by identifying 1 and -1 on the Riemann sphere, and let p be the point at ∞. Then V_n may be identified with the space of polynomials in the affine coordinate t of the sphere, of degree at most n, which have the same value at ±1: $V_n = \{f(t)$ of degree $\le n | f(1) = f(-1)\}$.

In particular we see that $\{1, f=t^2\}$ is a basis for V_2 and $\{1, f, g=t^3-t\}$ is a basis for V_3. Then $g^2 = t^6-2t^4+t^2 = f^3 - 2f^2 + f$; thus $y = g$ and $x = f - 2/3$ is a Weierstrass basis for E; we have $y^2 = x^3 - (1/3)x + (2/27)$.

Assume that E is a cuspidal rational curve, whose normalization is again the Riemann sphere, with 0 over the cusp and ∞ over p. In this case V_n may be identified with the polynomials in t of degree at most n, whose derivative at 0 is 0: $V_n = \{f(t)$ of degree $\le n | f'(0) = 0\}$. In this case $x = t^2$ and $y = t^3$ is a Weierstrass basis for E, and we have $y^2 = x^3$.

An equation of the form $y^2 = x^3 + Ax + B$ for an elliptic curve over $\mathbb{K}$ is called a <u>Weierstrass equation</u> for E over $\mathbb{K}$. The discriminant $\Delta = 4A^3 + 27B^2$ vanishes if and only if E is singular, and Δ is well-defined up to multiplication by 12^{th} powers in $\mathbb{K}$. The isomorphism class of E, if E is smooth, is determined by the J-function of A and B as defined in (I.2): $J(A,B) = 4A^3/\Delta$. The pair (A,B) are referred to as the <u>Weierstrass coefficients</u> of E, and by the Corollary above they are determined up to the given (λ^4, λ^6) action of $\mathbb{K}^*$.

If $\mathbb{K}$ is algebraically closed, we have three types of orbits of this action of $\mathbb{K}^*$ on $\mathbb{K}^2$:

- the orbit of (0,0): this is the cuspidal rational curve
- the orbit of (-3,2): this is the nodal rational curve
- the other orbits: these are the orbits of smooth elliptic curves.

II.3: Weierstrass fibrations.

We want to develop the theory of the Weierstrass equation for an elliptic surface, i.e., for a family of elliptic curves. Unfortunately a smooth elliptic surface does not have all of its fibers simply irreducible curves of arithmetic genus one: see Table (I.4.1). In addition, the Weierstrass equation is for a curve with a chosen point. Therefore we have to make two assumptions: that there is a section to the elliptic surface (i.e., a point chosen in every fiber), and that the fibers are all irreducible.

The first assumption, that of a section, is rather serious: you can't create one if there isn't one there already. The second assumption, though, is quite mild, if you are willing to live with a non-smooth surface.

Assume that $\pi: X \longrightarrow C$ is a smooth minimal elliptic surface with section S. (As we saw in section (III.1), the assumption of smooth and minimal is not serious.) Note that for every fiber F, $S \cdot F = 1$, so that S can meet only one component of any fiber, and that component must have multiplicity one. By considering the list of singular fibers, one sees that if F is of type I_N, II, III, or IV, then S can meet any one of the fiber components; if F is of type I_N^*, then S can meet only one of the 4 multiplicity one components. If F is of type IV^*, then S can meet only one of the 3 components with multiplicity 1; if F is of type III^*, then S can meet only one of the 2 components with multiplicity one; and if F has type II^*, there is only one component with multiplicity one, and S must meet it. If π has a section, then no fiber can be multiple, so no F can be of type ${}_M I_N$ with $M > 1$.

Consider a reducible fiber F of π. As the reader can check, the components of F not meeting S forms a connected set of smooth rational curves each with self-intersection -2, meeting with dual graph one of the Dynkin diagrams. The following table tells which diagram goes with which fiber type.

(II.3.1)<u>Table</u> of dual graphs for the reducible Kodaira fibers minus a component of multiplicity one.

<u>Kodaira fiber type</u>	<u>Dual graph of fiber minus a component with mult. 1</u>
I_N, $N \geq 2$	A_{N-1}
I_N^*	D_{N+4}
III	A_1
IV	A_2
IV^*	E_6
III^*	E_7
II^*	E_8

One obtains an irreducible fiber by contracting all the components of F not meeting the section S. This will give a singular surface if π has reducible fibers, but the singularities are quite mild: they are simply rational double points (RDP's) of the type denoted by the Dynkin diagram.

If one starts with a smooth minimal elliptic surface $\pi: X \longrightarrow C$ with section S, and performs these contractions, one will obtain an elliptic surface $\overline{\pi}: \overline{X} \longrightarrow C$, with section S, such that all fibers are irreducible. In this context we have a chance at a global Weierstrass equation.

Let us abstract this concept.

(II.3.2):Definition: Let X be a surface and Y be a smooth curve. A Weierstrass fibration $\pi:X \longrightarrow C$ is a flat and proper map π from X to C such that every geometric fiber has arithmetic genus one (i.e., is either a smooth genus one curve, a rational curve with a node, or a rational curve with a cusp), with general fiber smooth, and such that there is given a section S of π, not passing through the nodes or cusps of any fiber.

The above discussion shows that there is a map F from the set of isomorphism classes of smooth minimal elliptic surfaces over C with section S to isomorphism classes of Weierstrass fibrations over C, given by contracting all components of fibers not meeting S. There is also a map G from isomorphism classes of Weierstrass fibrations over C to smooth minimal elliptic surfaces over C with section, given by taking the unique smooth minimal model, which exists and is unique by Corollary (II.1.3). It is clear that $G \circ F$ is the identity on smooth minimal elliptic surfaces over C with section: F contracts the necessary components and G resolves them back again. Hence F is injective and G is surjective.

However they are not inverse: $F \circ G$ is not the identity in general. To see this, take a product surface $X = E \times C$, where (E,p) is elliptic; this is smooth and minimal (π is the projection onto C, of course) and a section is given by $S = \{p\} \times C$.

Choose a point c in C, and consider the point $x = (p,c)$ on X. Blow up x on X and blow down the proper transform of the fiber of π through x; this is a smooth elliptic curve with self-intersection -1, and so blowing it down produces a surface $\overline{X}$ with an elliptic singularity. The new fiber over c is now the image of the exceptional divisor over x, and this is a rational curve with a cusp. $\overline{X}$ is a Weierstrass fibration over C, with section S. Applying F to $\overline{X}$ however gives the product surface back again, and applying G to the product leaves the product unchanged, so in this case $G(F(\overline{X})) = X$.

We'll return to this problem later.

Getting back to the general theory, let $\pi:X \longrightarrow C$ be a Weierstrass fibration with section S. We have the exact sequence for the normal bundle of S in X:

(II.3.3) $$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(S) \longrightarrow N_{S/X} \longrightarrow 0.$$

Apply π_* to this sequence, and obtain

(II.3.4) $$0 \longrightarrow \pi_*\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_X(S) \longrightarrow \pi_*N_{S/X} \longrightarrow R^1\pi_*\mathcal{O}_X \longrightarrow R^1\pi_*\mathcal{O}_X(S) \longrightarrow 0.$$

(Note that $R^1\pi_*N_{S/X} = 0$ since $N_{S/X}$ is supported on S, and the fibers of π

restricted to S have dimension 0.)

In this situation the nicest possible base change results apply: we have that both $\pi_*\mathcal{O}_X(nS)$ and $R^1\pi_*\mathcal{O}_X(nS)$ are locally free for all n, and that $[\pi_*\mathcal{O}_X(nS)]|_c \cong H^0(X_c, \mathcal{O}_{X_c}(nS))$ and $[R^1\pi_*\mathcal{O}_X(nS)]|_c \cong H^1(X_c, \mathcal{O}_{X_c}(nS))$, where c is any point of C and X_c is the fiber of π over c. In particular, using Riemann-Roch on the fibers, we have that

(II.3.5) $\pi_*\mathcal{O}_X = \mathcal{O}_C$ and
$\pi_*\mathcal{O}_X(nS)$ is a locally free sheaf of rank n on C for all $n \geq 1$.

Moreover, $R^1\pi_*\mathcal{O}_X$ is a line bundle on C, and
$R^1\pi_*\mathcal{O}_X(nS) = 0$ for all $n \geq 1$.

Now let us return to the sequence (II.3.4). From the above statement, we have that the last term is zero; hence the map from $\pi_*N_{S/X}$ to $R^1\pi_*\mathcal{O}_X$ is a surjection. However $R^1\pi_*\mathcal{O}_X$ is a line bundle on C, and so is $\pi_*N_{S/X}$ ($N_{S/X}$ is a line bundle on S, and π is an isomorphism on S). Therefore this map must be an isomorphism:

(II.3.6) $\pi_*N_{S/X} \cong R^1\pi_*\mathcal{O}_X$.

This of course forces the map from $\pi_*\mathcal{O}_X(S)$ to $\pi_*N_{S/X}$ to be zero, and the first map in the sequence, from $\pi_*\mathcal{O}_X$ to $\pi_*\mathcal{O}_X(S)$, to be an isomorphism. Since π is proper, we have $\pi_*\mathcal{O}_X = \mathcal{O}_C$, so

(II.3.7) $\pi_*\mathcal{O}_X(S) \cong \pi_*\mathcal{O}_X = \mathcal{O}_C$.

II.4: The fundamental line bundle $\mathbb{L}$.

Let $\pi:X \longrightarrow C$ be a Weierstrass fibration with section S.The line bundle $\pi_*N_{S/X}$ is a fundamental invariant of the Weierstrass fibration. Let us give its inverse a name.

(II.4.1)<u>Definition:</u> Let $\pi:X \longrightarrow C$ be a Weierstrass fibration with section S. The <u>fundamental</u> <u>line</u> <u>bundle</u> of π is the line bundle $\mathbb{L} = [\pi_*N_{S/X}]^{-1}$ on C.

Note that because of (II.3.6), the fundamental line bundle does not depend on the section S which is given for π; it could equally well have been defined as the inverse to $R^1\pi_*\mathcal{O}_X$, which is obviously independent of S.

Since we have $\mathcal{O}_X((n-1)S) \subset \mathcal{O}_X(nS)$ for every n, we also have $\pi_*\mathcal{O}_X((n-1)S) \subset \pi_*\mathcal{O}_X(nS)$ for every n. By (II.3.5), these are both locally free sheaves if $n \geq 1$. They are equal (to $\mathcal{O}_C$) if $n = 1$, and for $n > 1$ we have the following relationship with the fundamental bundle $\mathbb{L}$.

(II.4.2)<u>Lemma:</u> There exists a short exact sequence

$$0 \longrightarrow \pi_*\mathcal{O}_X((n-1)S) \longrightarrow \pi_*\mathcal{O}_X(nS) \longrightarrow \mathbb{L}^{-n} \longrightarrow 0$$

for every $n \geq 2$.

Proof: Consider the exact sequence
$0 \to \mathcal{O}_X((n-1)S) \to \mathcal{O}_X(nS) \to \mathcal{O}_S(nS) \to 0$. Applying π_*, and noticing that if $n \geq 2$ then $R^1\pi_*\mathcal{O}_X((n-1)S) = 0$, we obtain the exact sequence
$0 \to \pi_*\mathcal{O}_X((n-1)S) \to \pi_*\mathcal{O}_X(nS) \to \pi_*\mathcal{O}_S(nS) \to 0$. Since $\mathcal{O}_S(nS) \cong N_{S/X}^{\otimes n}$, we are done. ■

Now the theory of the Weierstrass equation comes into play. In each fiber of the sheaf $\pi_*\mathcal{O}_X(3S)$ there is picked out canonically three natural directions, namely the spans of 1, x, and y, where (x,y) is any Weierstrass basis for that fiber. (Although the Weierstrass basis (x,y) is not determined, the span of x and y are, by Lemma (II.2.2); the span of 1 is exactly the subsheaf $\mathcal{O}_X$.) These directions give a basis at every fiber, and so give a <u>splitting</u> of $\pi_*\mathcal{O}_X(3S)$.

(II.4.3)<u>Lemma:</u> For every $n \geq 2$, we have a splitting

$$\pi_*\mathcal{O}_X(nS) \cong \mathcal{O}_X \oplus \mathbb{L}^{-2} \oplus \mathbb{L}^{-3} \oplus \dots \oplus \mathbb{L}^{-n}.$$

Proof: Locally, let (x,y) be a Weierstrass basis for the fiber of π. Then if $n = 2$, the splitting is given by the span of the elements 1 and x; if $n = 3$, by the span of 1, x, and y. If n is even, say $n = 2m$, then the splitting is given locally by the spans of $\{1,x,x^2,\dots,x^m,y,xy,\dots,x^{m-2}y\}$; if n is odd, say $n = 2m+1$, then the splitting is given locally by the spans of $\{1,x,x^2,\dots,x^m,y,xy,\dots,x^{m-1}y\}$. By the uniqueness of the Weierstrass basis (up to the given action of $\mathbb{C}^*$), these spans are canonically determined, and give a splitting of $\pi_*\mathcal{O}_X(nS)$.

To see that the splitting is as stated, just apply Lemma (II.4.2) and use induction. ■

II.5: Weierstrass data.

Fix a sufficiently fine open cover $\{\mathcal{U}_i\}$ for C, so that $\mathbb{L}$ is trivialized on each $\mathcal{U}_i$, and pick a basis section e_i on each $\mathcal{U}_i$ for $\mathbb{L}|_{\mathcal{U}_i}$. Then e_i^{-n} is a basis section for $\mathbb{L}^{-n}$ on $\mathcal{U}_i$. Assume that (α_{ij}) are the transition functions for $\mathbb{L}$ with respect to the local bases (e_i), i.e., $e_i = \alpha_{ij}e_j$ on $\mathcal{U}_i \cap \mathcal{U}_j$.

In constructing the Weierstrass basis locally over $\mathcal{U}_i$, we may begin by choosing an f_i in $\pi_* \mathcal{O}_X(2S)|_{\mathcal{U}_i}$ such that f_i projects onto e_i^{-2} in $\mathbb{L}|_{\mathcal{U}_i}$, and choosing a g_i in $\pi_* \mathcal{O}_X(3S)|_{\mathcal{U}_i}$ such that g_i projects onto e_i^{-3} in $\mathbb{L}_i|_{\mathcal{U}_i}$. The original equation for g_i^2, you will recall, is $g_i^2 = a_6f_i^3 + a_5f_ig_i + a_4f_i^2 + a_3g_i + a_2f_i + a_1$, where the a_i are sections of $\mathcal{O}_{U_i}$. This equation is an equality between two sections of $\pi_* \mathcal{O}_X(6S)|_{\mathcal{U}_i}$. Considering the projections onto the factor $\mathbb{L}^{-6}|_{\mathcal{U}_i}$, we see that the left hand side projects to e_i^{-6}, and the right hand side to $a_6e_i^{-6}$; hence we must have $a_6 = 1$.

A local Weierstrass basis (x_i, y_i) is obtained by completing the square in g_i (to get y_i), then completing the cube in f_i (to get x_i). Since f_i and g_i project to e_i^{-2} and e_i^{-3} in $\mathbb{L}^{-2}|_{\mathcal{U}_i}$ and $\mathbb{L}^{-3}|_{\mathcal{U}_i}$ respectively, and the process of completing the square and the cube only affects terms of lower order, we have that x_i and y_i also project to e_i^{-2} and e_i^{-3} in $\mathbb{L}^{-2}|_{\mathcal{U}_i}$ and $\mathbb{L}^{-3}|_{\mathcal{U}_i}$ respectively. Hence, as local generators for the direct summands of $\pi_* \mathcal{O}_X(3S)|_{\mathcal{U}_i}$ isomorphic to $\mathbb{L}^{-2}|_{\mathcal{U}_i}$ and $\mathbb{L}^{-3}|_{\mathcal{U}_i}$ respectively, they transform like the given local bases e_i^{-2} and e_i^{-3}. This proves the following.

(II.5.1) Lemma: For each i there is a local Weierstrass basis (x_i, y_i) which transform by $x_i = \alpha_{ij}^{-2}x_j$ and $y_i = \alpha_{ij}^{-3}y_j$.

Given the local Weierstrass basis (x_i, y_i), we obtain the local Weierstrass coefficients (A_i, B_i), which are locally sections of $\mathcal{O}_{U_i}$. We have, over $\mathcal{U}_i \cap \mathcal{U}_j$,

$$x_i^3 + A_ix_i + B_i = y_i^2 = \alpha_{ij}^{-6}y_j^2 = \alpha_{ij}^{-6}[x_j^3 + A_jx_j + B_j] = x_i^3 + \alpha_{ij}^{-4}A_jx_i + \alpha_{ij}^{-6}B_j,$$

showing that A_i and B_i transform by $A_i = \alpha_{ij}^{-4}A_j$ and $B_i = \alpha_{ij}^{-6}B_j$. Therefore the local sections $\{A_ie_i^4\}$ and $\{B_ie_i^6\}$ patch together to give <u>global</u> <u>sections</u> of $\mathbb{L}^4$

and $\mathbb{L}^6$ respectively: $A_i e_i^4 = A_j e_j^4$ and $B_i e_i^6 = B_j e_j^6$. We usually call these global sections simply A and B.

(II.5.2)<u>Definition:</u> The pair of sections (A,B) for $\mathbb{L}^4 \oplus \mathbb{L}^6$ are called the <u>Weierstrass coefficients</u> for the Weierstrass fibration $\pi: X \longrightarrow C$. The <u>discriminant</u> of the fibration is the section $\Delta = 4A^3 + 27B^2$ of $\mathbb{L}^{12}$.

(II.5.3)<u>Lemma:</u> Given a Weierstrass fibration $\pi: X \longrightarrow C$, the discriminant Δ is not identically zero (in particular the Weierstrass coefficients are not both zero); moreover the Weierstrass coefficients (A,B) are well-defined up to the action of λ in $H^0(C,\mathcal{O}_C)^*$ given by $\lambda\cdot(A,B) = (\lambda^4 A, \lambda^6 B)$.

Proof: If the discriminant is identically zero, then every fiber of the Weierstrass fibration would be singular, which is not allowed. The uniqueness statement follows from the local uniqueness (which is essentially Corollary (II.2.3)). ■

Notice then that the discriminant Δ is well-defined up to multiplication by a 12^{th} power in $H^0(C,\mathcal{O}_C)^*$.

We now have all the data collected which can be obtained from a Weierstrass fibration, namely, the fundamental line bundle $\mathbb{L}$, and the Weierstrass coefficients (A,B).

(II.5.4)<u>Definition:</u> A triple $(\mathbb{L},A,B)$ will be called <u>Weierstrass data</u> for a Weierstrass fibration over C, or simply Weierstrass data over C, if $\mathbb{L}$ is a line bundle on C and (A,B) are global sections of $\mathbb{L}^4 \oplus \mathbb{L}^6$ such that the section $\Delta = 4A^3 + 27B^2$ of $\mathbb{L}^{12}$ (which is called the <u>discriminant</u> of the data) is not identically 0.

We have shown that if $\pi: X \longrightarrow C$ is a Weierstrass fibration over C, then the triple $(\mathbb{L},A,B)$, where $\mathbb{L} = [R^1\pi_*\mathcal{O}_X]^{-1}$ and (A,B) are Weierstrass coefficients for $\pi: X \longrightarrow C$, is Weierstrass data over C. Moreover, the fiber of π over a point c of C is singular if and only if the discriminant section Δ is zero at c.

Conversely, given Weierstrass data $(\mathbb{L},A,B)$ over C, the Weierstrass fibration is built by patching together the local surfaces defined by the local Weierstrass equations $y_i^2 = x_i^3 + A_i x_i + B_i$, where the A_i and B_i are the local versions of A and B over a sufficiently fine open cover $\{\mathcal{U}_i\}$ for C.

The uniqueness statement of Lemma (II.5.3) shows that Weierstrass data for $\pi:X \longrightarrow C$ is unique up to isomorphism, where we say two sets of Weierstrass data $(\mathbb{L}_1,A_1,B_1)$ and $(\mathbb{L}_2,A_2,B_2)$ are isomorphic if there is an isomorphism of line bundles $\phi:\mathbb{L}_1 \longrightarrow \mathbb{L}_2$, inducing isomorphisms $\phi^4:\mathbb{L}_1^4 \longrightarrow \mathbb{L}_2^4$ and $\phi^6:\mathbb{L}_1^6 \longrightarrow \mathbb{L}_2^6$, such that ϕ^4 transports A_1 to A_2 and ϕ^6 transports B_1 to B_2. (If $\mathbb{L}_1 = \mathbb{L}_2$, then ϕ must be multiplication by an element in $H^0(C,\mathcal{O}_C)^*$, and we recover the uniqueness statement above.)

Of course we say two Weierstrass fibrations $\pi_1:X_1 \longrightarrow C$ and $\pi_2:X_2 \longrightarrow C$ are isomorphic if there is an isomorphism of surfaces $f:X_1 \longrightarrow X_2$ such that $\pi_1 = \pi_2 \circ f$. In this way we have a 1-1 correspondence:

$$\text{(II.5.5)} \qquad \left\{\begin{matrix}\text{Weierstrass}\\ \text{data}\\ \text{over C}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Weierstrass}\\ \text{fibrations}\\ \text{over C}\end{matrix}\right\}$$

(both sets up to isomorphism).

The reader will see now why we chose the "fundamental" bundle $\mathbb{L}$ to be the inverse of $R^1\pi_*\mathcal{O}_X$ rather than the R^1 itself: $\mathbb{L}$ tends to be a nonnegative bundle.

(II.5.6)<u>Lemma:</u> Let $(\mathbb{L},A,B)$ be Weierstrass data over a projective curve C. Then $\deg(\mathbb{L}) \geq 0$. Moreover, if $\deg(\mathbb{L}) = 0$, then either $\mathbb{L}^4$ or $\mathbb{L}^6$ is trivial. Hence if $\deg(\mathbb{L}) = 0$, $\mathbb{L}$ is torsion in Pic(C) of order 1, 2, 3, 4, or 6.

Proof: The bundle $\mathbb{L}^{12}$ has a non-zero section, so its degree must be non-negative. This proves the first statement, and the second follows from the condition that at least one of A or B is not zero. ∎

(II.5.7)<u>Lemma:</u> The number of singular fibers of a Weierstrass fibration over a projective curve C is 12 times the degree of the fundamental line bundle, counting properly.

Of course this is just the degree of the discriminant divisor $\{\Delta = 0\}$ on C. We see in this context that "counting properly" means that a singular fiber counts for the number of zeroes of Δ.

Lecture III: <u>The global aspects of the Weierstrass equation.</u>

III.1: The representation as a divisor in a $\mathbb{P}^2$- bundle.

Let $\pi:X \longrightarrow C$ be a Weierstrass fibration, with section S. Recall (Lemma (II.4.3)) that $\pi_* \mathcal{O}_X(3S) \cong \mathcal{O}_C \oplus \mathbb{L}^{-2} \oplus \mathbb{L}^{-3}$, where $\mathbb{L}$ is the fundamental line bundle of π. Let $\phi:\pi^*\pi_*\mathcal{O}_X(3S) \longrightarrow \mathcal{O}_X(3S)$ be the natural map. Note that, locally on C, $\mathcal{O}_X(3S)$ is generated by 1, x, and y (using the notation of the previous lecture), and that these are exactly the local generators for $\mathcal{O}_C$, $\mathbb{L}^{-2}$, and $\mathbb{L}^{-3}$; hence this natural map ϕ is a surjection of $\mathcal{O}_X$-modules. Since $\mathcal{O}_X(3S)$ is a line bundle on X, this gives a map $f:X \longrightarrow \mathbb{P}(\pi_*\mathcal{O}_X(3S))$; since $\pi_*\mathcal{O}_X(3S)$ is a locally free rank 3 sheaf on C, $\mathbb{P} = \mathbb{P}(\pi_*\mathcal{O}_X(3S))$ is a $\mathbb{P}^2$-bundle over C. Moreover, if p is the structure map from $\mathbb{P}$ to C, we have $p \circ f = \pi$.

In fact, it is clear that f is an embedding: it is on each fiber, since 1, x, and y generate the homogeneous coordinate ring of every fiber of π, by the Weierstrass equation analysis. Therefore, via f, we have realized X as a divisor inside the $\mathbb{P}^2$-bundle $\mathbb{P}$ over C.

Let (A,B) be the Weierstrass coefficients of X over C. A global equation for X in $\mathbb{P}$ is then given by $Y^2Z = X^3 + AXZ^2 + BZ^3$. This simply means that such an equation describes X locally over C, if A and B are interpreted as local functions on C by suitably choosing a local trivializing section for $\mathbb{L}$ (hence for $\mathbb{L}^4$ and $\mathbb{L}^6$). The given section is defined by $X = Z = 0$. The global variables (Z,X,Y) are interpreted as formal sections of $(\mathcal{O}_C,\mathbb{L}^2,\mathbb{L}^3)$ to transform properly (Lemma (II.5.1)).

This implies that the divisor class of X in $\mathbb{P}$ is $(p^*\mathbb{L}^6)(3)$: the individual terms of the global equation are sections in $\mathbb{L}^6$ formally, and the equation is of degree 3 in the global variables.

The canonical class of $\mathbb{P}$ is $(p^*(\omega_C \otimes \mathbb{L}^{-5}))(-3)$ (in general, one pulls back the canonical class of C tensored with the determinant of the bundle, then twists by $\mathcal{O}_{\mathbb{P}}(-\text{rank})$). Hence the adjunction bundle of $\mathbb{P}$ to X is $p^*(\omega_C \otimes \mathbb{L})$, so we may use the adjunction formula to deduce the following.

(III.1.1)<u>Proposition:</u> $\omega_X \cong \pi^*(\omega_C \otimes \mathbb{L})$.

In particular, note that the canonical divisor on X is pulled back from the curve C, so that

(III.1.2) $K_X^2 = 0$.

Also, the growth of $H^0(nK_X)$ can never be more than linear, so that

(III.1.3) the Kodaira dimension of X is at most 1.

Also clear is the following.

(III.1.4)<u>Lemma:</u> X is a product of C with a smooth elliptic curve if and only if $\mathbb{L} \cong \mathcal{O}_C$.

Proof: Indeed, if X is a product, then I claim that the normal bundle to any section must be trivial, so $\mathbb{L} \cong \mathcal{O}_C$. This follows essentially because an elliptic curve is a group. Let (E,e_0) be an elliptic curve, and let $f:C \longrightarrow E$ induce a section S of $X = E \times C$, by $S = \{(f(c),c)|c \in C\}$. Apply the automorphism σ_f to X, defined by $\sigma_f(e,c) = (e-f(c),c)$. It is clear that σ_f carries S into the "horizontal" section $\{(e_0,c)\}$, which has trivial normal bundle since of course it is a fiber of the projection to E.

Conversely, if $\mathbb{L} \cong \mathcal{O}_C$, then $\mathbb{P} \cong \mathbb{P}^2 \times C$, and A and B must be constants, so X is a product. ■

(III.1.4)<u>Example:</u> Let $\pi:X \longrightarrow \mathbb{P}^1$ arise from a pencil of plane cubics. The last exceptional divisor for the blowup from $\mathbb{P}^2$ can be taken to be the given section, and so we see that $\mathbb{L} = \mathcal{O}_{\mathbb{P}^1}(1)$. Then $K_X = \pi^*\mathcal{O}(-1) = -F$, where F is a fiber of π. Note that here the Kodaira dimension is $-\infty$.

(III.1.5)<u>Example:</u> Let $C = \mathbb{P}^1$, and let $\mathbb{L} = \mathcal{O}_{\mathbb{P}^1}(2)$. Then $K_X = 0$, and we will see in a moment that X is a K3 surface. In any case, it is clear that here the Kodaira dimension is 0. If we take $\mathbb{L} = \mathcal{O}_{\mathbb{P}^1}(k)$ with $k \geq 3$, then K_X will be a positive multiple of the fiber, and will move in at least a pencil; the Kodaira dimension will be 1.

It will be useful to remark the following:

(III.1.6) $$H^0(C,\mathbb{L}^{-1}) = \begin{cases} 0 & \text{if X is \underline{not} a product} \\ 1 & \text{if X \underline{is} a product} \end{cases}.$$

Indeed, the second statement is clear using Lemma (III.1.4), and the first follows because $\mathbb{L}$ has non-negative degree.

III.2: The representation as a double cover of a ruled surface.

Let $R = \mathbb{P}(\pi_* \mathcal{O}_X(2S)) = \mathbb{P}(\mathcal{O}_C \oplus \mathbb{L}^{-2})$; R is a ruled surface over C, and we will call q the structure map from R to C. The natural surjection from $\pi^* \pi_* \mathcal{O}_X(2S)$ to $\mathcal{O}_X(2S)$ gives a map $g: X \longrightarrow R$, which is a C-map: $q \circ g = \pi$. Locally, this map is given by sending (x,y) to x, and is therefore a double covering, branched over the trisection T defined in R by $X^3 + AXZ^2 + BZ^3 = 0$ and the section given by $Z = 0$. Note that in R, the trisection T and the section $Z = 0$ are disjoint.

The involution of the double cover g is simply $(x,y) \longmapsto (x,-y)$, and so we see that this is exactly the inverse map on the fibers of π. Therefore R is the quotient of X by the inverse map on the fibers, and we sometimes write this as $R = X/(\pm 1)$.

The trisection T is a divisor on R with line bundle $(q^* \mathbb{L}^6)(3)$, and the section $Z = 0$ has line bundle $\mathcal{O}_R(1)$, so the branch locus of g is a divisor with line bundle $(q^* \mathbb{L}^6)(4)$. The standard theory of double covers implies that $g_* \mathcal{O}_X \cong \mathcal{O}_R \oplus (q^* \mathbb{L}^{-3})(-2)$; the second factor is the sub-line bundle of $g_* \mathcal{O}_X$ locally generated by y.

Of course, when T intersects a fiber of R in 3 distinct points, the corresponding fiber of X is a smooth elliptic curve, and conversely. The singular fibers of π occur exactly over those fibers of R which T does not meet properly. Indeed, as we will see below, the local behaviour of the curve T on R completely determines the type of singular fiber of X.

III.3: Weierstrass data in minimal form

Recall from Lecture II that we have defined two maps back and forth between the set of isomorphism classes of smooth minimal elliptic surfaces over C with section, and the set of isomorphism classes of Weierstrass fibrations over C:

$$\left\{\begin{matrix}\text{smooth minimal}\\ \text{elliptic surfaces}\\ \text{over C w/section}\end{matrix}\right\} \underset{G}{\overset{F}{\rightleftarrows}} \left\{\begin{matrix}\text{Weierstrass}\\ \text{fibrations}\\ \text{over C}\end{matrix}\right\}$$

where F contracts all components of fibers not meeting the section, and G resolves any singularities of the Weierstrass fibration, then blows down any (-1)-curves in the fibers to get a relatively minimal surface. (Equivalently, $G(\pi: X \longrightarrow C)$ is the unique smooth minimal birational model, see Corollary (II.1.3).)

As explained in (II, section 3), F is an inclusion, G a surjection, and

$G \circ F$ is the identity. However $F \circ G$ is not the identity, and there are plenty of Weierstrass fibrations which are not "hit" by F.

(III.3.1)<u>Definition:</u> A Weierstrass fibration $\pi : X \longrightarrow C$ is <u>in minimal form</u> if it is hit by F, i.e., if $F(G(\pi)) = \pi$.

We naturally want to analyze the Weierstrass fibrations in minimal form: they are in one-to-one correspondence with the smooth minimal elliptic surfaces over C with section, hence with elliptic curves over K(C).

The following Proposition gives a satisfying answer to this problem. First some notation: a triple tacnode of a curve T on a smooth surface is a triple point t of T, such that t is still a triple point on the blowup (an infinitely near triple point, or two consecutive triple points). The order of vanishing of a section s of a line bundle at a point c will be denoted by $\nu_c(s)$.

(III.3.2)<u>Proposition:</u> Let $\pi : X \longrightarrow C$ be a Weierstrass fibration over C, with Weierstrass data $(\mathbb{L}, A, B)$. The following are equivalent.

(a) π is in minimal form.

(b) X has only rational double points (RDPs) as singularities.

(c) The trisection T of the ruled surface $R = X/(\pm 1)$ has no triple tacnodes.

(d) There is no point c in C where $\nu_c(A) \geq 4$ and $\nu_c(B) \geq 6$.

Proof: That (a) implies (b) is clear, by considering the classification of the singular fibers; see Table (II.3.1). That (b) and (c) are equivalent follows from the general theory of double coverings. If $g : X \longrightarrow R$ is a double cover of a smooth surface, with branch locus D, then X is smooth if and only if D is smooth, and X has only RDPs if and only if D has no points of multiplicity greater than 3, and has no triple tacnodes. In our case D has two disjoint components T and the section $Z = 0$, and the section is certainly smooth, so the criterion need only be applied to T; since T is a trisection (meeting the fibers of R three times), T can have no point of multiplicity 4 or more. Hence T has no triple tacnodes if and only if X has only RDPs.

The reader can easily check that (c) implies (d): if t is a local coordinate at c on C, and t^4 divides A and t^6 divides B, then the curve $X^3 + A(t)X + B(t) = 0$ has a triple tacnode at $(x,t) = (0,0)$. Conversely, assume that $X^3 + A(t)X + B(t) = 0$ has a triple tacnode at a point $(x_0, 0)$. In particular, the cubic $X^3 + A(0)X + B(0)$ has a triple root; this implies (since there is no square term) that the triple root must be at $X = 0$, i.e., $A(0) = B(0) = 0$, and the triple tacnode must occur at $(0,0)$. For $X^3 + A(t)X + B(t)$ to have a triple point at $(0,0)$, we must have $t^2|A$ and $t^3|B$. If we now blow up the origin, replacing X by Xt and dividing by t^3, we obtain $X^3 + (A(t)/t^2)X + (B(t)/t^3)$, and this must have a triple point also. The same argument shows then that $t^2|A(t)/t^2$ and $t^3|B(t)/t^3$, so that $t^4|A$ and $t^6|B$: this is the denial of condition (d).

We now have that (b), (c), and (d) are equivalent, and that (a) implies them; we must show that they imply (a). Assume that $\pi:X \longrightarrow C$ is not in minimal form. In this case X certainly cannot be smooth; pick a fiber where X is not smooth, and work locally at this fiber (all the conditions are local on C). Let $\alpha:\tilde{X} \longrightarrow X$ be a minimal resolution of X. If $\tilde{X}$ is minimal over C, then α must simply be the contraction of all of the components of the singular fiber not meeting the section, so that X is in the image of F, and is minimal. Hence we may assume that $\tilde{X}$ is not minimal over C, and let $\beta:\tilde{X} \longrightarrow Y$ be the minimilization. The map β must contract the proper transform of the original irreducible fiber of π, since if it does not, Y would dominate X and $\tilde{X}$ would not be a minimal resolution.

Now view the process going the other way, starting from Y. In order to get to X, we must at some stage blow up the point of intersection of the section with the fiber of Y, else β would not contract the proper transform of the original fiber of π. After performing a certain number of such blowups, we arrive at $\tilde{X}$, and to get to X we contract all other components of the fiber of $\tilde{X}$ except that last blowup made at the intersection point with the section. In this case what is being blown down is at least the entire fiber of Y; hence, since this has arithmetic genus one, the singularity on X cannot be an RDP. This violates condition (b). ∎

To preserve the uniformity of notation, we will say that Weierstrass data $(\mathbb{L},A,B)$ over C is _in minimal form_ if for every c in C, either $\nu_c(A) \le 3$ _or_ $\nu_c(B) \le 5$.

We will refer to the process F∘G (first taking the unique smooth minimal model, then contracting components not meeting the section) as _putting the Weierstrass fibration into normal form._ We would like to describe what this means, algorithmically, in terms of the Weierstrass data.

Let $(\mathbb{L},A,B)$ be Weierstrass data over C. For each c in C, define $n_c = \max\{n \ge 0 | \nu_c(A) \ge 4n \text{ and } \nu_c(B) \ge 6n\}$. Consider the divisor $D = \sum_{c\in C} n_c c$ on C. Let f be a section of $\mathcal{O}_C(D)$ vanishing on exactly D.

(III.3.4)<u>Lemma:</u> The Weierstrass data for the normal form of $(\mathbb{L},A,B)$ is $(\mathbb{L}(-D),A/f^4,B/f^6)$.

Proof: By the maximality of n_c for each c, the Weierstrass data $(\mathbb{L}(-D),A/f^4,B/f^6)$ is in normal form, and of course by the definition of the n_c's, A/f^4 is holomorphic as is B/f^6, and they are sections of $\mathbb{L}(-D)^{\otimes 4}$ and $\mathbb{L}(-D)^{\otimes 6}$ respectively. We must only check that the Weierstrass fibrations given by $(\mathbb{L}(-D),A/f^4,B/f^6)$ and $(\mathbb{L},A,B)$ are birational. This follows from considering the general fibers, with their Weierstrass fibrations, and applying Corollary (II.2.3), using for $\lambda \in K(C)$ the meromorphic function corresponding to f. ∎

III.4: **Invariants of a Weierstrass fibration**

I'd like in this section to compute the cohomology groups $H^i(X,\mathcal{O}_X)$ for a Weierstrass fibration, and also to compute the plurigenera of X, as well as the other standard invariants.

First let us address the irregularity q of X; it is at least g (the genus of the base curve C) and I claim that unless X is a product, this is exactly q: all 1-forms on X are pulled back from C.

(III.4.1) $$q = \begin{cases} g & \text{if X is \underline{not} a product} \\ g+1 & \text{if X \underline{is} a product} \end{cases}.$$

Proof: The easiest way to see this is to use the short exact sequence of low-order terms in the Leray spectral sequence for the map π. This sequence has $E_2^{pq} = H^p(C,R^q\pi_*\mathcal{O}_X)$, abutting to $H^{p+q}(X,\mathcal{O}_X)$. The relevant terms of the short exact sequence are $0 \to E_2^{10} \to H^1(X,\mathcal{O}_X) \to E_2^{01} \to E_2^{20}$, which for us is $0 \to H^1(C,\pi_*\mathcal{O}_X) \to H^1(X,\mathcal{O}_X) \to H^0(C,R^1\pi_*\mathcal{O}_X) \to H^2(C,\pi_*\mathcal{O}_X)$. Since $\pi_*\mathcal{O}_X \cong \mathcal{O}_C$, we have that the last term is zero (C is a curve) and the first term has dimension g. The third term is simply $H^0(C,\mathbb{L}^{-1})$, and so the result follows using Lemma (III.1.6). ∎

(III.4.2) $$p_g = \begin{cases} g + \deg(\mathbb{L}) - 1 & \text{if X is } \underline{\text{not}} \text{ a product} \\ g + \deg(\mathbb{L}) & \text{if X } \underline{\text{is}} \text{ a product} \end{cases}.$$

Proof: This is a more straightforward calculation:

$p_g = h^2(X,\mathcal{O}_X) = h^0(X,\omega_X)$ (Serre duality on X)

$= h^0(X,\pi^*(\omega_C\otimes\mathbb{L}))$ (using (III.1.1))

$= h^0(C,\omega_C\otimes\mathbb{L})$ (from the projection formula)

$= h^1(C,\mathbb{L}^{-1})$ (Serre duality on C)

$= h^0(\mathbb{L}^{-1}) - \chi(\mathbb{L}^{-1}) = h^0(\mathbb{L}^{-1}) - \deg(\mathbb{L}^{-1}) - 1 + g$ (using Riemann-Roch)

$= g + \deg(\mathbb{L}) - 1 + h^0(\mathbb{L}^{-1})$,

which proves the result using Lemma (III.1.6). ■

This then gives a succinct formula for $\chi(\mathcal{O}_X)$:

(III.4.3): $\chi(\mathcal{O}_X) = \text{degree}(\mathbb{L})$.

Using Noether's formula, with the fact that $K_X^2 = 0$, now gives:

(III.4.4): $e(X) = 12\cdot\text{degree}(\mathbb{L})$.

The plurigenera of X are easy to calculate, also. Since we have already computed p_g, let us assume that $n \geq 2$. Then

$P_n(X) = h^0(X,\omega_X^{\otimes n}) = h^0(X,\pi^*(\omega_C\otimes\mathbb{L})^{\otimes n})$ (Proposition III.1.1)

$= h^0(C,\omega_C^n\otimes\mathbb{L}^n)$ (the projection formula)

$= \chi(\omega_C^n\otimes\mathbb{L}^n) + h^1(\omega_C^n\otimes\mathbb{L}^n)$

$= \deg(\omega_C^n\otimes\mathbb{L}^n) + 1 - g + h^1(\omega_C^n\otimes\mathbb{L}^n)$ (using Riemann-Rich on C)

$= n(2g-2+\deg(\mathbb{L})) + 1 - g + h^0(\omega_C^{1-n}\otimes\mathbb{L}^{-n})$ (using Serre duality).

This seems a convenient form for computations, and it gives the following with ease.

(III.4.5)<u>Lemma:</u> Assume that n is at least two.

(a) Let $g = 0$. Then

$$P_n(X) = \begin{cases} 0 & \text{if } \deg(\mathbb{L}) \le 1 \\ 1+n(k-2) & \text{if } \deg(\mathbb{L}) = k \ge 2. \end{cases}$$

(b) Let $g = 1$. If $\deg(\mathbb{L}) = 0$, recall that then $\mathbb{L}$ is torsion of order $t = 1$, 2, 3, 4, or 6. In this case

$$P_n(X) = \begin{cases} 1 & \text{if } t \text{ divides } n \\ 0 & \text{otherwise} \end{cases}.$$

If $\deg(\mathbb{L}) \ge 1$, then $P_n(X) = n\cdot\deg(\mathbb{L})$.

(c) Let $g \ge 2$. Then $P_n(X) = n(2g - 2 + \deg(\mathbb{L})) + 1 - g$.

These calculations allow us to place elliptic surfaces with section in the over-all classification of surfaces.

(III.4.6)<u>Lemma:</u>

(a) Let $g = 0$. Then X is
- a product $E \times \mathbb{P}^1$ if $\deg(\mathbb{L}) = 0$,
- a rational surface if $\deg(\mathbb{L}) = 1$,
- a K3 surface if $\deg(\mathbb{L}) = 2$, and
- a properly elliptic surface if $\deg(\mathbb{L}) \ge 3$.

(b) Let $g = 1$. Then X is
- an abelian surface (a product) if $\mathbb{L} \cong \mathcal{O}_C$.
- a hyperelliptic ("bielliptic" in Beauville's notation) surface if $\mathbb{L}$ is torsion of order 2, 3, 4, or 6, and
- a properly elliptic surface if $\deg(\mathbb{L}) \ge 1$.

In case X is hyperelliptic, the order of K_X is that of $\mathbb{L}$.

(c) Let $g \ge 2$. Then X is a properly elliptic surface.

Lecture IV: The Local Aspects of the Weierstrass Equation.

1: A bit more on invariants and classification.

There are just a few more remarks I want to make along the lines of the last section of the previous lecture, that is, on the invariants and classification of Weierstrass fibrations, or elliptic surfaces with section. Firstly, what we have computed up to now allows us to compute all the Hodge numbers.

(IV.1.1) Lemma: Let $\pi: X \longrightarrow C$ be a smooth minimal elliptic surface over C with section, with associated line bundle $\mathbb{L}$. If X is not a product surface, then the Hodge diamond of X is

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ g+\deg(\mathbb{L})-1 & & 10\deg(\mathbb{L})+2g & & g+\deg(\mathbb{L})-1. \\ & g & & g & \\ & & 1 & & \end{array}$$

In particular, $h^{1,1} = 10\deg(\mathbb{L}) + 2g$. If X is a product $E \times C$, then the Hodge diamond of X is

$$\begin{array}{ccccc} & & 1 & & \\ & g+1 & & g+1 & \\ g+\deg(\mathbb{L}) & & 10\deg(\mathbb{L})+2g+2 & & g+\deg(\mathbb{L}). \\ & g+1 & & g+1 & \\ & & 1 & & \end{array}$$

In particular, $h^{1,1} = 10\deg(\mathbb{L}) + 2g + 2$.

Proof: All of the hodge numbers except for $h^{1,1}$ can be gotten immediately from (III.4.1) and (III.4.2). Then, since the alternating sum of the Hodge numbers is the Euler number e(X), $h^{1,1}$ can be readily found using (III.4.4). ■

Note that if X is a rational surface, then the genus g of C must be zero, i.e., C must be isomorphic to $\mathbb{P}^1$. (This follows from (III.4.1), or from Lemma (III.4.6).) In fact, from Lemma (III.4.6), if X is rational, then $g = 0$ and $\mathbb{L} \cong \mathcal{O}_{\mathbb{P}^1}(1)$. We have seen in Lecture I many examples of rational elliptic surfaces constructed from a pencil of cubic curves in $\mathbb{P}^2$, and all of course have base curve $\mathbb{P}^1$ and $\deg(\mathbb{L}) = 1$. (The last exceptional curve for the blow-up from $\mathbb{P}^2$ is a section S, with $S^2 = -1$; hence $\deg(\mathbb{L}) = 1$.) The converse is also true.

(IV.1.2)Lemma: Let $\pi:X \longrightarrow \mathbb{P}^1$ be a rational elliptic surface with section. Then X is the 9-fold blowup of the plane $\mathbb{P}^2$ at the base points of a pencil of generically smooth cubic curves which induces the fibration π.

Proof: Let $f:X \longrightarrow M$ be the blow-down of X to a minimal model M. Note that because $K_X = -F$, where F is a fiber (Proposition (III.1.1)), every smooth rational curve E on X has $E^2 \geq -2$. Therefore M cannot be $\mathbb{F}_k$ with $k \geq 3$; the section on $\mathbb{F}_k$ with self-intersection $-k$ would persist on X to a smooth rational curve E with $E^2 \leq -k \leq -3$. Hence M is either $\mathbb{F}_0$, $\mathbb{F}_2$, or $\mathbb{P}^2$. If M is $\mathbb{F}_0$, then any blow-up of M also dominates $\mathbb{P}^2$; if M is $\mathbb{F}_2$, then no point of the (-2)-section may be blown up to get to X (since then the proper transform of the (-2)-section would be a smooth rational curve E with $E^2 \leq -3$), and any blow-up of $\mathbb{F}_2$ at a point not on the (-2)-section also dominates $\mathbb{P}^2$. Therefore we may assume M is the plane $\mathbb{P}^2$. The pencil $|F|$ on X descends to a pencil on $\mathbb{P}^2$, and since F is $-K_X$, and the general member of F is smooth, we have that the image pencil in $\mathbb{P}^2$ is a pencil of generically smooth curves in $|-K_{\mathbb{P}^2}|$, i.e., is a pencil of generically smooth cubic curves. ∎

There is an interesting relationship here between rational elliptic surfaces with section and weak Del Pezzo surfaces of degree 1. Given a rational elliptic surface X with section S, the surface Y obtained from X by blowing down S is a weak Del Pezzo of degree 1. Conversely, given a weak Del Pezzo surface Y of degree one, the pencil $|-K_Y|$ has a unique base point, which upon being blown up yields a rational elliptic surface with section. Hence:

(IV.1.3)Lemma: There is a one-to-one correspondence between

$$\left\{\begin{array}{l}\text{rational elliptic}\\ \text{surfaces over } \mathbb{P}^1\\ \text{with section}\end{array}\right\} \text{ and } \left\{\begin{array}{l}\text{Weak Del Pezzo}\\ \text{surfaces}\\ \text{of degree 1}\end{array}\right\}.$$

2: The singularities of the trisection T

In preparation for an analysis of the local information to be gotten from the Weierstrass coefficients (A,B) of a Weierstrass fibration (or, from a smooth minimal elliptic surface with section) we will analyze the singular fibers of the fibration from the point of view of the singularities of the trisection T of the ruled surface R of which the Weierstrass fibration is a double cover, as described in (III, section 2). For definiteness let $\pi:X \longrightarrow C$

be a smooth minimal elliptic surface over C with section, and let Y be the associated Weierstrass fibration in minimal form.

Recall that T is a trisection of the ruling on R, and the branch locus of the double covering is T union the section $\{Z = 0\}$. Fix a fiber F of R; the singular fiber of π is determined by the local behaviour of T near F. In fact, very little geometric information about T is needed to determine the Kodaira type of the fiber of π over F, as we will see.

By the minimality of the Weierstrass fibration, T has no triple tacnodes. The analytic types of a curve singularity with no triple tacnodes (and no points of multiplicity 4 or more) are classified, and have no moduli: they are the so-called simple curve singularities, also sometimes referred to as the a-d-e curve singularities. They are amply discussed in [BPV, Chapter II, section 8], and I'll just briefly present the facts in the table below.

(IV.2.1)Table of simple curve singularities, of a curve on a smooth surface. Here C is the curve with singularity at p, E is the exceptional divisor after blowing up p, and $\overline{C}$ is the proper transform of C on the blow-up.

Name	Local Equation	Geometric Description
a_0	$x = 0$	smooth point of C
a_1	$x^2 = y^2$	ordinary node
a_2	$x^2 = y^3$	ordinary cusp
a_n	$x^2 = y^{n+1}$	higher-order cusp or tacnode, if $n \geq 3$
d_4	$yx^2 = y^3$	ordinary triple point
$d_{n\geq 5}$	$yx^2 = y^{n-1}$	triple point of C, with $\overline{C}$ meeting E in two points, one smooth and one singular of type a_{n-5}. (If $n = 5$, this means that $\overline{C}$ is tangent to E there.)
e_6	$x^3 = y^4$	triple point of C with one tangent, such that $\overline{C}$ is smooth and meets E at one point to order 3.
e_7	$x^3 = xy^3$	triple point of C with one tangent, such that $\overline{C}$ has an ordinary node (type a_1) with E as one tangent.
e_8	$x^3 = y^5$	triple point of C with one tangent, such that $\overline{C}$ has an ordinary cusp (type a_2) with E as tangent.

The Kodaira fiber type of the elliptic fibration π is determined only be the singularity of T, and the relative position of the fiber of R with respect to the tangents to T at the singular points; this follows from an analysis of the double covering, and is quite elementary. In particular, we have the following.

(IV.2.2)<u>Proposition:</u> Let T be the trisection of R, F a fiber of R, and G the fiber of π over F.

(a) Assume $T|_F = p + q + r$, with p, q, and r distinct points of F. Then G has type I_0, i.e., G is a smooth elliptic curve.

(b) Assume $T|_F = p + 2q$, with p and q distinct points of F. Then q is at worst a double point of T, and if it is double, then F is not one of the tangents. Moreover:

(b1) If T is smooth at q, then G has type I_1, i.e., is a nodal rational curve.

(b2) If T has a double point at q, of type a_{n-1}, then G has type I_n.

(c) Assume $T|_F = 3p$. Then p is at worst a triple point of T.

(c1) If T is smooth at p, then G has type II, i.e., is a cuspidal rational curve.

(c2) If T is double at p, of type a_n, then F must be one of the tangents to T at p; hence $n \leq 2$ (else $(T\cdot F)_p \geq 3$).

(c2i): If (T,p) is of type a_1, then G has type III.

(C2ii): If (T,p) is of type a_2, then G has type IV.

(c3) If T is triple at p, then F is not one of the tangents.

(c3i): If (T,p) is of type d_n, then G has type I^*_{n-4}.

(c3ii): If (T,p) is of type e_6, then G has type IV^*.

(c3iii): If (T,p) is of type e_7, then G has type III^*.

(c3iv): If (T,p) is of type e_8, then G has type II^*.

Proof: Most of the statements follow from the classification of the fibers, Table (II.3.1), and the well-known fact that the double cover of a smooth surface germ branched over a curve with a simple singularity is a rational double point, and the type of the RDP has the same name as the type of the curve singularity. I.e., if T has a singular point of type a_n, then the double cover has an RDP of type A_n, and similarly for the d_n's and e_n's. (See, for example, [BPV, Chapter III, section 7].)

Part (a) is clear; and part (c3) follows from the argument above. Most of part (b) also follows; if T has a singularity of type a_n with $n \geq 3$ then there is no ambiguity to the Kodaira type, by Table (II.3.1). We must only check that if T has a point of types a_0, a_1, or a_2, then the position of the fiber as described above determines the type of G as indicated.

In other words, we must distinguish between I_1 and II, between I_2 and

III, and between I_3 and IV. This can be done by remarking that for the I_n fibers, the fiber of the Weierstrass fibration is a nodal rational curve; hence the trisection T must meet the fiber in two points, not one. For the types II, III, and IV, the fiber of the Weierstrass fibration is a cuspidal rational curve; hence trisection T must meet the fiber in only one point. This completes the analysis. ■

3: The a, b, δ table

The point of making the previous Proposition is to notice that very little geometric information about T is required to determine the Kodaira type of the fiber G. We want to use these remarks to develop "Tate's algorithm" for determining the Kodaira type of the fiber G from the Weierstrass coefficients (A,B), and the discriminant $\Delta = 4A^3 + 27B^2$. Fix a point c on the base curve c, and denote by a, b, and δ the order of vanishing of A, B, and Δ respectively, at c: $a = \nu_c(A)$, $b = \nu_c(B)$, and $\delta = \nu_c(\Delta)$. Then I claim that the Kodaira type is completely determined by the three integers a, b, and δ. Note that if we assume minimality of the Weierstrass data, then we have $a \le 3$ <u>or</u> $b \le 5$. The algorithm is presented in the Table below, along with some extra information which will be useful. The legend for the table is:

LEGEND for Table (IV.3.1):

Name:	Kodaira's name for the type of singular fiber
Curve:	A description of the curve itself (often using the dual graph)
a:	the order of vanishing of A
b:	the order of vanishing of B
δ:	the order of vanishing of $\Delta = 4A^3 + 27B^2$
J:	the value of the J-function
m(J):	the multiplicity of the J-function
e:	the euler number of the fiber
r:	the number of components of the fiber not meeting the section
d:	the number of components of the fiber with multiplicity one
RDP:	the RDP obtained by contracting components not meeting S_0 (Also, the type of singularity of the branch curve T)
γ:	the genus drop contributed by the singularity of T

(IV.3.1)Table

Name	Curve	a	b	δ	J	m(J)	e	r	d	RDP	γ	Comments
I_0	smooth elliptic curve	0 a≥1 0	0 0 b≥1	0 0 0	≠0,1,∞ 0 1	? 3a 2b	0	0	1	-	0	T meets F in three points
I_1	nodal rational curve	0	0	1	∞	1	1	0	1	-	0	T is tangent to F
I_N	cycle of N smooth rational curves	0	0	N	∞	N	N	N-1	N	A_{N-1}	[N/2]	T meets F twice at the double point
I_0^*	$\tilde{D}_4$	2 a≥3 2	3 3 b≥4	6 6 6	≠0,1,∞ 0 1	? 3a-6 2b-6	6	4	4	D_4	3	T has an ordinary triple point on F
I_N^*	$\tilde{D}_{N+4}$	2	3	N+6	∞	N	N+6	N+4	4	D_{N+4}	3+[N/2]	T has a point of type (3,2) on F
II	cuspidal rational curve	a≥1	1	2	0	3a-2	2	0	1	-	0	T is flexed to F at one point
III	two tangent rational curves	1	b≥2	3	1	2b-3	3	1	2	A_1	1	T has node with F as one tangent
IV	three concurrent rational curves	a≥2	2	4	0	3a-4	4	2	3	A_2	1	T has cusp with F as tangent
IV^*	$\tilde{E}_6$	a≥3	4	8	0	3a-8	8	6	3	E_6	3	$\bar{T}$ is flexed to E
III^*	$\tilde{E}_7$	3	b≥5	9	1	2b-9	9	7	2	E_7	4	$\bar{T}$ has a node on E with one tangent E
II^*	$\tilde{E}_8$	a≥4	5	10	0	3a-10	10	8	1	E_8	4	$\bar{T}$ has a cusp on E with E as tangent

In the last column, $\bar{T}$ is the proper transform of T after one blow-up, and E is the first exceptional curve.

Proof: We'll start by simply verifying that if a, b, and δ are as in the table, then the Kodaira type of the fiber of π is as indicated. Again preserve the notation of Proposition (IV.2.2): F is the fiber of the ruled surface, and G is the fiber of the elliptic surface. G is smooth if and only if $\delta = 0$, and G is of type I_n, with $n \geq 1$, if and only if $\delta \neq 0$ and $a = b = 0$; this last statement follows from the remark that only the I_n fibers with $n \geq 1$

have a nodal rational curve for the Weierstrass fibration, and if $a = b = 0$, then the Weierstrass fibration has a cusp; also, one should note that if $\delta \neq 0$, then $a = 0$ if and only if $b = 0$. This already proves the first line of the table, i.e., that for I_0.

Let us take up the second and third lines, that for the I_N fibers with $N \geq 1$. We must only check now that if $a = b = 0$, then we have an I_δ fiber, assuming $\delta \geq 1$. By Proposition (IV.2.2) it suffices to check that T has a double point of type $a_{\delta-1}$ at the point where T meets F twice. By scaling appropriately, we may assume that T has the equation $x^3 + (-3 + f(t))x + (2 + g(t))$ for local functions f and g with $f(0) = g(0) = 0$. The discriminant is computed to be $\Delta = 108(f+g)-36f^2+4f^3+27g^2$. The singular point of T occurs at $(x,t) = (1,0)$, and the other branch of solutions passes through $(x,t) = (-2,0)$. By changing coordinates analytically we may suppose that the other branch is exactly $x = -2$, and so T has a local equation near $(1,0)$ of the form $x^2 + (-2 + \alpha)x + (1 + \beta)$, for some local functions α and β with $\alpha(0) = \beta(0) = 0$; the discriminant in terms of α and β is then $\Delta = \alpha^2 - 4(\alpha + \beta)$. By replacing x by $z + 1 - \alpha/2$, we have the equation$z^2 + (\alpha + \beta - \alpha^2/4) = z^2 - \Delta/4$. Hence, using (IV.2.1), we see that this is a local equation for a simple singularity of type $a_{\delta-1}$, and so we have G of type I_δ as required.

The analysis for the I_N^* cases are very similar and I will leave them to the reader. Let us just present the rest of the analysis quickly. Note that we have $a \geq 1$ and $b \geq 1$ for the rest of the cases, and T meets F at only one point p, which is $(x,t) = (0,0)$ in the Weierstrass equation $x^3+A(t)x+B(t)$.

Assume that G has type II. Then by Proposition (IV.2.2), case (c1), T is smooth at p, and so we must have $b = 1$.

Assume that G has type III. Then by case (c2i), T has an ordinary node at $(0,0)$, with one tangent $\{t=0\}$; hence $a = 1$ and $b \geq 2$.

Assume that G has type IV. Then by case (c2ii), T has an ordinary cusp at $(0,0)$, with tangent $\{t=0\}$; hence $b = 2$ and $a \geq 2$.

In the rest of the cases T has a triple point at $(0,0)$, with no tangent F (which is $\{t=0\}$ at this point), so $a \geq 3$ and $b \geq 4$. Write $A = t^a\alpha$ and $B = t^b\beta$ with $\alpha(0)$ and $\beta(0)$ nonzero. Upon blowing up $(0,0)$, we obtain the equation $x^3 + t^{a-2}\alpha x + t^{b-3}\beta$, with exceptional divisor defined by $t = 0$, and to distinguish between the last three cases we must decide whether this equation describes a smooth curve (giving e_6 and IV^*), a curve with a node (giving e_7 and III^*), or a curve with a cusp (giving e_8 and II^*). This is smooth if and only if $b = 4$; it has a node if and only if $a = 3$ and $b \geq 5$; and it has a cusp if and only if $b = 5$ and $a \geq 5$. This finishes the verification of the first part of the table.

I will leave to the reader to check that the J-function is as indicated; this is a straightforward calculation using only the definition of J. The Euler number e of the fiber is also easily calculated; most of the fibers have all $\mathbb{P}^1$s for components, and e is then directly related to the number of components. The columns for r and d need no explanation, and the genus drop γ is as indicated, using only that the genus drop for a double point is one and for a triple point is 3. ∎

Some easy remarks are in order now, which are obtained simply by inspection of the table.

(IV.3.2)<u>Lemma:</u>

(a) In all cases $e = \delta$.

(b) In all cases $0 \leq e - r \leq 2$, with $e - r = \begin{cases} 0 & \text{if type } I_0\text{, i.e. smooth} \\ 1 & \text{if type } I_N \text{ with } N \geq 1 \\ 2 & \text{otherwise} \end{cases}$.

We say that a fiber is <u>semistable</u> if is of type I_N for some $N \geq 0$. The above invariant $e - r$ then distinguishes between semistable and non-semistable fibers.

Note that the equality $e = \delta$ is a local version of the equality $e(X) = \text{degree}(\Delta)$ (both are equal to $12 \cdot \deg(\mathbb{L})$, by (III.4.4)). Of course, $\text{degree}(\Delta)$ is the sum of the local δ's, and also the Euler number $e(X)$ of the smooth surface X is the sum of the local Euler numbers, since we have a fibration by genus 0 (hence $e = 0$) curves:

(IV.3.3)<u>Lemma:</u> Let $\pi: X \longrightarrow C$ be a smooth minimal elliptic surface. Then $e(X) = \sum_c e(\pi^{-1}(c))$.

Proof: Let $S \subset C$ be the finite set of points of C over which the fiber of π is singular, and denote by X_c the fiber of π over $c \in C$. Then

$$e(X) = e(\pi^{-1}(C - S)) + \sum_{c\in S} e(X_c)$$
$$= e(\text{a smooth elliptic curve})\cdot e(\pi^{-1}(C - S) + \sum_{c\in S} e(X_c)$$
$$= \sum_{c\in S} e(X_c) \text{ since a smooth elliptic curve has Euler number 0.} \blacksquare$$

4: The J-function

Let us turn to the J-function, and what we can derive from Table (IV.3.1). Again simply by inspection, we have the following.

(IV.4.1)<u>Lemma:</u> Let G be a fiber of π over $c \in C$.

(a) If G has type II, IV, IV^*, or II^* then $J(c) = 0$.
Assume $J(c) = 0$. Then:
G has type I_0 or I_0^* if and only if $m(J) \equiv 0 \bmod 3$
G has type II or IV^* if and only if $m(J) \equiv 1 \bmod 3$.
G has type IV or II^* if and only if $m(J) \equiv 2 \bmod 3$.

(b) If G has type III or III^* then $J(c) = 1$.
Assume $J(c) = 1$. Then:
G has type I_0 or I_0^* if and only if $m(J) \equiv 0 \bmod 2$.
G has type III or III^* if and only if $m(J) \equiv 1 \bmod 2$.

(c) G has type I_N or I_N^* with $N \geq 1$ if and only if J has a pole at c of order N.

From the last statement, and remarking that J the poles of J can occur only at singular fibers, we have a formula for the degree of the J-function.

It is convenient to use the following notation: let i_N (respectively i_N^*, ii, iii, iv, iv^*, iii^*, ii^*) denote the number of fibers of type I_N (respectively I_N^*, II, III, IV, IV^*, III^*, II^*); similarly let $i_0(j)$ (respectively $i_0^*(j)$) denote the number of smooth fibers (respectively fibers of type I_0^*) with $J = j$.

(IV.4.2)<u>Corollary:</u> Considering J as a map from C to $\mathbb{P}^1$, we have

$$\text{degree}(J) = \sum_{N\geq 1} N(i_N + i_N^*).$$

We need to analyze the Hurwitz formula as it applies to the J-map, to obtain an important inequality. Assume that the J-function is not constant.

For each j in $\mathbb{P}^1$ and positive integer m define integers $k_j(m)$ by

$$k_j(m) = \#\{\ c \in C \mid J(c) = j \text{ and } \text{mult}_c(J) = m\ \}.$$

Let d = degree(J). Then

(IV.4.3): $$d = \sum_{m\geq 1} mk_j(m) \quad \text{for each fixed } j \text{ in } \mathbb{P}^1.$$

Using Lemma (IV.4.1), we have the following relationships.

(IV.4.4)
$$\begin{aligned}
i_0(0) + i_0^*(0) &= \sum_{m\equiv 0(3)} k_0(m)\\
ii + iv^* &= \sum_{m\equiv 1(3)} k_0(m)\\
iv + ii^* &= \sum_{m\equiv 2(3)} k_0(m)\\
i_0(1) + i_0^*(1) &= \sum_{m\equiv 0(2)} k_1(m)\\
iii + iii^* &= \sum_{m\equiv 1(2)} k_1(m)\\
i_n + i_n^* &= k_\infty(n)
\end{aligned}$$

A bit more subtle are the following inequalities.

(IV.4.5)<u>Lemma:</u>

(a) $i_0(0) + i_0^*(0) \leq [d - (ii + iv^*) - 2(iv + ii^*)]/3$,

and the right hand side is a non-negative integer.

Moreover, equality holds if and only if

(Condition J0): every fiber of type II or IV* has m = 1,
every fiber of type IV or II* has m = 2, and
every fiber of type I_0 and I_0^* with J = 0 has m = 3.

I.e., if J(c) = 0, then $\text{mult}_c(J) \leq 3$.

(b) $i_0(1) + i_0^*(1) \leq [d - (iii + iii^*)]/2$,

and the right hand side is a non-negative integer.

Moreover, equality holds if and only if

(Condition J1): every fiber of type III or III* has m = 1, and
every fiber of type I_0 or I_0^* with J = 1 has m = 2.

I.e., if J(c) = 1 then $\text{mult}_c(J) \leq 2$.

Proof: From the previous observations (IV.4.4), we have

$$(ii + iv^*) + 2(iv + ii^*) + 3(i_0(0) + i_0^*(0)) = \sum_{m\equiv 1(3)} k_0(m) + 2\sum_{m\equiv 2(3)} k_0(m) + 3\sum_{m\equiv 0(3)} k_0(m)$$

$\le d$, and is congruent to d modulo 3, by (IV.4.3), applied with $j = 0$. Statement (a) follows from this; statement (b) is similar:

$$(iii + iii^*) + 2((i_0(1) + i_0^*(1)) = \sum_{m\equiv 1(2)} k_1(m) + 2\sum_{m\equiv 0(2)} k_1(m) \le d, \text{ and is}$$

congruent to d modulo 2. ■

Let R_j denote the ramification contributed by the points c of C with $J(c) = j$; we have

(IV.4.6) $$R_j = \sum_{m\ge 1} (m-1)k_j(m) = d - \sum_{m\ge 1} k_j(m).$$

The previous lemma can be brought to bear to give a good estimate on R_0 and R_1.

(IV.4.7) Lemma:

(a) $R_0 \ge [2d - 2(ii + iv^*) - (iv + ii^*)]/3$.
Moreover, the right hand side is an integer, and equality holds if and only if condition J0 of Lemma (IV.4.5) holds.

(b) $R_1 \ge [d - (iii + iii^*)]/2$.
Moreover, the right hand side is an integer, and equality holds if and only if condition J1 of Lemma (IV.4.5) holds.

(c) $R_\infty = d - \sum_{m\ge 1} (i_m + i_m^*)$.

Proof: Statement (c) is a direct consequence of (IV.4.6) and (IV.4.4), applied with $j = \infty$. For part (a), we have

$$R_0 = d - \sum_{m\ge 1} k_0(m) = d - \sum_{m\equiv 1(3)} k_0(m) + \sum_{m\equiv 2(3)} k_0(m) + \sum_{m\equiv 0(3)} k_0(m)$$

$$= d - (ii + iv^*) - (iv + ii^*) - (i_0(0) + i_0^*(0))$$

$$\ge d - (ii + iv^*) - (iv + ii^*) - [d - (ii + iv^*) - 2(iv + ii^*)]/3$$

(using the previous lemma)

$$= [2d - 2(ii + iv^*) - (iv + ii^*)]/3 \text{ as required.}$$

Part (b) is similar:

$$R_1 = d - \sum_{m\geq 1} k_1(m) = d - \sum_{m\equiv 1(2)} k_1(m) + \sum_{m\equiv 0(2)} k_1(m)$$
$$= d - (iii + iii^*) - (i_0(1) + i_0^*(1))$$
$$\geq d - (iii + iii^*) - [d - (iii + iii^*)]/2$$
$$= [d - (iii + iii^*)]/2 \text{ as indicated.}$$

The other statements follow from their counterparts in the previous lemma. ■

We are finally ready to apply the Hurwitz formula to the J-map.

(IV.4.8)Proposition: Assume that J is not constant. Let
$$x = 2g - 2 + \frac{1}{6}\left[6\sum_{m\geq 1} (i_m + i_m^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - d\right].$$
Then x is a non-negative integer. Moreover, if x = 0 then conditions J0 and J1 hold, and the only ramification of the J-map occurs over 0, 1, and ∞.

Proof: Let $R' = \sum_{j\neq 0,1,\infty} R_j$ be the total ramification away from 0, 1, and ∞. Then the Hurwitz formula for J gives

$$2g - 2 = -2d + R_0 + R_1 + R_\infty + R'$$
$$\geq -2d + R_0 + R_1 + R_\infty \quad \text{(with equality if and only if } R' = 0)$$
$$\geq -2d + [2d - 2(ii + iv^*) - (iv + ii^*)]/3 + [d - (iii + iii^*)]/2$$
$$+ [d - \sum_{m\geq 1} (i_m + i_m^*)] \quad \text{(with equality if and only if conditions J0 and J1 hold)}$$
$$= d/6 - \sum_{m\geq 1} (i_m + i_m^*) - 2(ii + iv^*)/3 - (iv + ii^*)/3 - (iii + iii^*)/2,$$

and re-arranging gives the result. ■

The integer x represents the "extra" ramification of the J-map, not forced by the particular configuration of singular fibers. When there is no "extra" ramification, the ramification of J is determined by the singular fibers. This happens, by the above Proposition, if and only if conditions J0 and J1 hold. We will say that an elliptic surface <u>has no extra J-ramification</u> if and only if J is not constant and x = 0. Thus:

(IV.4.9)Corollary: An elliptic surface $\pi: X \longrightarrow C$ has no extra J-ramification if and only if J is not constant, the only branch points of J are 0,1, and ∞, and
- o whenever J(c) = 0, $\mathrm{mult}_c(J) \leq 3$,
- o whenever J(c) = 1, $\mathrm{mult}_c(J) \leq 2$.

Lecture V: <u>The J-map.</u>

1: Criterion for birationality

In this section, which could (maybe should) have come much before this, I'd like to briefly present the criterion for determining when two sets of Weierstrass data $(\mathbb{L}_1,A_1,B_2)$ and $(\mathbb{L}_2,A_2,B_2)$ determine birational surfaces. An effective procedure is obtained by Lemma (III.3.4), Corollary (II.1.3), and the notion of isomorphism for Weierstrass data (the discussion after Definition (II.5.4): one puts both sets of data into minimal form, and then checks whether the data are then isomorphic.

Another way to say this is as follows.

(V.1.1)<u>Proposition:</u> Two sets of Weierstrass data $(\mathbb{L}_1,A_1,B_2)$ and $(\mathbb{L}_2,A_2,B_2)$ determine birational surfaces if and only if there are line bundles M_1, M_2 on C and sections $f_1 \in H^0(M_1)$, $f_2 \in H^0(M_2)$ such that $(\mathbb{L}_1 \otimes M_1, A_1f_1^4, B_1f_1^6) \cong (\mathbb{L}_2 \otimes M_2, A_2f_2^4, B_2f_2^6)$.

Proof: Clearly if such line bundles and sections exist, the two sets of data define birational surfaces. Conversely, if the two sets of data define isomorphic surfaces, then when brought to normal form they become isomorphic. I.e., there are two divisors D_1 and D_2 on C, and sections g_1, g_2 of $\mathcal{O}_C(D_1)$, $\mathcal{O}_C(D_2)$ respectively, such that $(\mathbb{L}_1(-D_1), A_1/g_1^4, B_1/g_1^6) \cong (\mathbb{L}_2(-D_2), A_2/g_2^4, B_2/g_2^6)$. Tensoring both sides of this isomorphism with $\mathcal{O}_C(D_1+D_2)$, and multiplying the A_i and B_i by $g_1^4g_2^4$ and $g_1^6g_2^6$ respectively, we see that, letting $M_1 = \mathcal{O}_C(D_2)$ and $M_2 = \mathcal{O}_C(D_1)$, $f_1 = g_2$ and $f_2 = g_1$, proves the result. ■

We will abuse language and say that the two sets of Weierstrass data $(\mathbb{L}_1,A_1,B_2)$ and $(\mathbb{L}_2,A_2,B_2)$ are <u>birational</u> if they determine birational surfaces. It is obvious that birationality of Weierstrass data is an equivalence relation; let $\mathbb{BW}$ denote the birational equivalence classes. Note that by the uniqueness of minimal models, and the correspondence between minimal models and Weierstrass data in minimal form, we may identify $\mathbb{BW}$ with the set of Weierstrass data in minimal form, up to isomorphism.

2: Quadratic twists and criterion for having the same J-map

Of course, if two sets of Weierstrass data $(\mathbb{L}_i, A_i, B_i)$ define birational surfaces, one expects the J-maps for the two sets to be the same, and this follows from the previous Proposition: the J-map is determined by the Weierstrass coefficients (A,B), and $J(A,B) = J(Af^4, Bf^6)$, since f^{12} factors out of both the numerator and denominator of the formula for J.

Actually, one can do a bit better than this. Note that $J(A,B) = J(Af^2, Bf^3)$; f^6 factors out of the numerator and denominator now. There is a converse to this calculation, which is as follows.

(V.2.1)<u>Proposition:</u> Let $(\mathbb{L}_1, A_1, B_1)$ and $(\mathbb{L}_2, A_2, B_2)$ be Weierstrass data over C. Let J_i be the J-map for $(\mathbb{L}_i, A_i, B_i)$, and assume that neither J_1 nor J_2 is identically 0 or 1. Then the following are equivalent:

(a) $J_1 = J_2$.

(b) There exists line bundles M_1 and M_2 on C and nonzero sections $f_i \in H^0(M_i^2)$, such that $(\mathbb{L}_1 \otimes M_1, A_1 f_1^2, B_1 f_1^3) \cong (\mathbb{L}_2 \otimes M_2, A_2 f_2^2, B_2 f_2^3)$.

Proof: That (b) implies (a) was remarked above. Assume then that $J_1 = J_2$ as maps from C to $\mathbb{P}^1$. Since neither J_i is identically 0 or 1, none of A_1, A_2, B_1, or B_2 are identically 0. The assumption that $J_1 = J_2$ then implies, after cross-multiplying and canceling some terms, that $A_1^3 B_2^2 = A_2^3 B_1^2$. Let $M_i = \mathbb{L}_1^3 \otimes \mathbb{L}_2^3 \otimes \mathbb{L}_i^{-1}$; let $f_1 = A_1 B_2$ and f_2 be $A_2 B_1$, and note that f_i are sections of M_i^2, and are nonzero by the hypotheses on the J_i. Then

$$(\mathbb{L}_1 \otimes M_1, A_1 f_1^2, B_1 f_1^3) = (\mathbb{L}_1^3 \otimes \mathbb{L}_2^3, A_1^3 B_2^2, A_1^3 B_1 B_2^3) = (\mathbb{L}_1^3 \otimes \mathbb{L}_2^3, A_2^3 B_1^2, A_2^3 B_1^3 B_2)$$
$$= (\mathbb{L}_2 \otimes M_2, A_2 f_2^2, B_2 f_2^3). \quad \blacksquare$$

This motivates the following definition. We will say that one has <u>performed a quadratic twist</u> on the Weierstrass data $(\mathbb{L}, A, B)$ if one replaces $(\mathbb{L}, A, B)$ by $(\mathbb{L} \otimes M, Af^2, Bf^3)$, where M is a line bundle on C and f is a nonzero section of M^2. The previous proposition states that, unless J is identically 0 or 1, Weierstrass data with the same J-maps are equal "up to quadratic twists".

Note a special case of the quadratic twist operation, namely when $M^2 \cong \mathcal{O}_C$, i.e., when M is a torsion line bundle of order 2 on C. Then we can take f = 1, and replace $(\mathbb{L}, A, B)$ by $(\mathbb{L} \otimes M, A, B)$. For example, when C is an elliptic curve, $\mathbb{L} \cong \mathcal{O}_C$, and A and B are nonzero, when we perform such a quadratic twist on the data $(\mathbb{L},A,B)$ (which represents a product of C with an elliptic curve with $J \neq 0$ or 1) we obtain (M,A,B), which represents a hyperelliptic surface with $K^{\otimes 2} \cong \mathcal{O}_C$ (see Lemma (III.4.6)).

Finally note that if we perform a quadratic twist <u>twice</u> with the same pair (M,f), we change $(\mathbb{L},A,B)$ to $(\mathbb{L}\otimes M^2, Af^4, Bf^6)$. Although these sets of data are different, by Proposition (V.1.1) they represent <u>birational</u> elliptic surfaces. This suggests that, in the proper setting, we have a group action here, and in the next section we will introduce the group involved.

3: The double cover group

Let C be a curve (not necessarily compact) and let S be an arbitrary subset of C. A <u>double cover pair</u> on C relative to S is a pair (M,f), where M is a line bundle on C and f is a nonzero section of $M^{\otimes 2}$, whose zero locus is contained (as a set) inside S.

The reason for this terminology is that if $\phi: D \longrightarrow C$ is a double covering (a flat finite map of degree 2) then $\phi_*\mathcal{O}_D$ is a vector bundle of rank 2 on C, which splits canonically as $\phi_*\mathcal{O}_D \cong \mathcal{O}_C \oplus M^{-1}$; M^{-1} is locally generated by the elements of trace zero, and a generator z of M^{-1} would satisfy an equation of the form $z^2 = f$, for some f in $\mathcal{O}_C$, locally. The multiplication in the $\mathcal{O}_C$-algebra $\phi_*\mathcal{O}_D$ is then defined by a map from $M^{-1}\otimes M^{-1}$ to $\mathcal{O}_C$, which locally of course simply sends $z\otimes z = z^2$ to the element f. This map can be viewed as a section (also rightly called f) of the bundle M^2, and is nonzero if we assume that D is reduced. This gives the double cover pair (M,f) associated to the covering ϕ. The requirement that (M,f) is a double cover pair relative to a subset S of C simply means that the branch locus of ϕ is contained inside S.

Two double cover pairs (M_1,f_1) and (M_2,f_2) are <u>isomorphic</u> if the is an isomorphism α between M_1 and M_2, inducing the isomorphism α^2 between M_1^2 and M_2^2, such that α^2 transports f_1 to f_2. Let $[M,f]$ denote the isomorphism class of a double cover pair (M,f), and let $\mathcal{A}_S$ be the set of isomorphism classes of double cover pairs over C relative to S.

One can define a product on the set $\mathcal{A}_S$ by declaring

$$[M_1,f_1]\cdot[M_2,f_2] = [M_1\otimes M_2, f_1f_2].$$

One easily checks that this product is well-defined on $\mathcal{A}_S$, and is associative and commutative, with the 2-sided identity element $[\mathcal{O}_C,1]$. Hence we have a monoid structure on the set $\mathcal{A}_S$.

Define a subset $\mathcal{B}_S$ of $\mathcal{A}_S$ to be the set of isomorphism classes $[M,f^2]$, where f is a nonzero section of M, whose zero set is inside S. It is easy to see that $\mathcal{B}_S$ is a submonoid of $\mathcal{A}_S$.

We wish to take the quotient of $\mathcal{A}_S$ by $\mathcal{B}_S$. In the category of monoids, the construction is as follows. Define an equivalence relation $\approx$ on $\mathcal{A}_S$ by declaring, for a_i in $\mathcal{A}_S$, $a_1 \approx a_2$ if and only if there are elements b_1 and b_2 of $\mathcal{B}_S$ such that $a_1 \cdot b_1 = a_2 \cdot b_2$. I.e., we declare $[M_1, f_1] \approx [M_2, f_2]$ if and only if there are line bundles N_1 and N_2 on C and nonzero sections g_i of N_i (whose zero loci are inside S), such that $[M_1 \otimes N_1, f_1 g_1^2] = [M_2 \otimes N_2, f_2 g_2^2]$.

The reader can check easily that $\approx$ is an equivalence relation on $\mathcal{A}_S$. Denote by $\{M,f\}$ the equivalence class of an element $[M,f]$ of $\mathcal{A}_S$.

(V.3.1)<u>Definition:</u> Let $\mathrm{Doub}_S(C)$ denote the set of equivalence classes $\{M,f\}$ of $\mathcal{A}_S$ modulo $\mathcal{B}_S$. If S = C, (so that there is no restriction on the zeroes of the sections involved), we denote $\mathrm{Doub}_C(C)$ by simply Doub(C).

The product on $\mathcal{A}_S$ now descends to a product on the set of $\approx$-classes $\mathrm{Doub}_S(C)$ (the reader should check that this is well-defined) which is of course still commutative, associative, with identity $\{\mathcal{O}_C, 1\}$. I claim in fact that now we have a group: every element of $\mathrm{Doub}_S(C)$ has order 2.

This is an easy check: $\{M,f\} \cdot \{M,f\} = \{M^2, f^2\}$, and f is a section of M^2, so $\{M,f\}^2$ is in $\mathcal{B}_S$; hence it is equal to $\{\mathcal{O}_C, 1\}$ in $\mathrm{Doub}_S(C)$. Thus:

(V.3.2)<u>Proposition:</u> $\mathrm{Doub}_S(C)$ is an abelian group, with every element having order two. The identity of $\mathrm{Doub}_S(C)$ is $\{\mathcal{O}_C, 1\}$.

In terms of double coverings, by factoring out $\mathcal{B}_S$ we have essentially identified covers which are birational; any cover defined by an element of $\mathcal{B}_S$ is globally split (it is defined by $z^2 = f^2$, not only locally but also globally) and hence any such cover is birational to the trivial cover (its normalization is isomorphic to the trivial cover). The product on $\mathcal{A}_S$ can be described as follows. Suppose a_1 and a_2 are in $\mathcal{A}_S$. representing covers $\phi_i : D_i \to C$. On any double cover one has the covering involution (sending z to -z if the cover is defined by $z^2 = f$) and so we have two covering involutions σ_1 and σ_2 on D_1 and D_2 respectively. The fiber product $D_1 \times_C D_2$ inherits both involutions, and they commute. The quotient $D_3 = (D_1 \times_C D_2)/(\sigma_1 \times \sigma_2)$ is a natural double cover of C, and its double cover pair represents the product of a_1 and a_2 in $\mathcal{A}_S$.

Since any element of $\mathcal{B}_S$ is birationally the trivial cover, any product of an element a of $\mathcal{A}_S$ and an element of $\mathcal{B}_S$ will be birationally the same as the cover represented by a. The converse is also true (although I won't spend time proving it here: it is easy) and so $\mathrm{Doub}_S(C)$ represents birational classes of double covers of C, whose branch locus is contained inside S.

Since we are talking about curves here, we may view $\mathrm{Doub}_S(C)$ as classifying double covers of C which are smooth: there is always a unique smooth model of the cover. Since a smooth double covering has a reduced branch locus, we suspect the following to be true (and it is!).

(V.3.3)<u>Lemma:</u> Every element of $\mathrm{Doub}_S(C)$ has a unique representative [M,f] in $\mathcal{A}_S$ with the divisor of zeroes $(f)_0$ of f reduced.

Proof: Let $\{M_1, f_1\}$ be in $\mathrm{Doub}_S(C)$, and write $(f)_0 = D + 2E$, with D reduced, and both D and E nonnegative. Let s be a section of $\mathcal{O}_C(E)$ which vanishes exactly along the divisor E, i.e., $(s)_0 = E$. Let $M = M_1(-E)$, and let $f = f_1/s^2$; note that by the construction of D and E, f is a holomorphic section of $\mathcal{L}^2$ and $(f)_0 = D$. Since $[M,f]\cdot[\mathcal{O}_C(E), s^2] = [M_1, f_1]$, and $[\mathcal{O}_C(E), s^2]$ is in $\mathcal{B}_S$, [M,f] also represents $\{M_1, f_1\}$ in $\mathrm{Doub}_S(C)$, and $(f)_0 = D$ is reduced.

This proves the existence, and let us turn to the uniqueness. Assume that $\{M_1, f_1\} = \{M_2, f_2\}$ with $D_1 = (f_1)_0$ and $D_2 = (f_2)_0$ both reduced. Then there are two line bundles N_1 and N_2 on C, and nonzero sections g_i of N_i, such that $[M_1 \otimes N_1, f_1 g_1^2] = [M_2 \otimes N_2, f_2 g_2^2]$. Let E_i be the divisor of zeroes of g_i: $E_i = (g_i)_0$. Then we have that $M_1 \otimes N_1 \cong M_2 \otimes N_2$ and $D_1 + 2E_1 = D_2 + 2E_2$. Since the D_i's are reduced, and all D_i's and E_i's are nonnegative, we must in fact have $D_1 = D_2$ and $E_1 = E_2$ (there is a unique way to decompose a nonnegative divisor as D+2E with D reduced and both D and E nonnegative). Since $E_1 = E_2$, we must have $N_1 \cong N_2$, and this forces $M_1 \cong M_2$. We may choose the isomorphism between the N_i so that the g_i correspond, since they have the same divisor; after tensoring the isomorphism guaranteed by $[M_1 \otimes N_1, f_1 g_1^2] = [M_2 \otimes N_2, f_2 g_2^2]$ with this isomorphism, we see that the isomorphism between the M_i's can be chosen to make the f_i's correspond. Hence $[M_1, f_1] = [M_2, f_2]$. ■

It is not hard to calculate the order of $\mathrm{Doub}_S(C)$ if S is finite and C is compact.

(V.3.4)<u>Lemma:</u> Suppose C is a complete curve. Then

$$|\mathrm{Doub}_S(C)| = \begin{cases} 2^{2g} & \text{if S is empty} \\ 2^{2g+|S|-1} & \text{if S is finite and nonempty} \end{cases}.$$

Proof: We need to count the double cover pairs (M,f) with $(f)_0$ reduced and contained inside S, by the previous Lemma. Since f is a section of M^2, the degree of $(f)_0$ must be even; thus the number of possible divisors $(f)_0$ is 1 if S is empty, and is $\sum_{i\geq 0} \binom{|S|}{2i} = 2^{|S|-1}$ if S is not empty.

Every such divisor determines the line bundle M^2, and so the number of possible choices for M is 2^{2g}, the number of 2-torsion elements in Pic(C). Once M and the divisor of zeroes of f are known, f is determined (up to isomorphism). ■

Now let us tie this double cover group construction in with the Weierstrass data. Let (M,f) be a double cover pair, and let $(\mathbb{L},A,B)$ be Weierstrass data. Note that the double cover pair information is exactly what is required to perform a quadratic twist on the Weierstrass data: the double cover pair (M,f) "twists" the data $(\mathbb{L},A,B)$ to the data $(\mathbb{L}\otimes M, Af^2, Bf^3)$.

Recall the set $\mathbb{BW}$ of birational classes of Weierstrass data. Denote the birational class of Weierstrass data $(\mathbb{L},A,B)$ by $\{\mathbb{L},A,B\}$. By the uniqueness of minimal models and the correspondence between minimal models and Weierstrass data in minimal form, every element $\{\mathbb{L},A,B\}$ of $\mathbb{BW}$ is represented by Weierstrass data in minimal form, uniquely up to isomorphism.

(V.3.5)<u>Proposition:</u> For any $S \subset C$, the above twisting operation induces a free action of $\mathrm{Doub}_S(C)$ on $\mathbb{BW}$. I.e., the action is given by

$$\{M,f\}\cdot\{\mathbb{L},A,B\} = \{\mathbb{L}\otimes M, Af^2, Bf^3\}.$$

Proof: First one must check that the twisting operation descends to the set of birational classes $\mathbb{BW}$; this is clear using Proposition (V.1.1). Moreover, if two double cover pairs are isomorphic, they act identically on $\mathbb{BW}$ (in fact they act so that the resulting data are isomorphic, not just birational); hence the operation induces an action of the monoid $\mathcal{A}_S$ on $\mathbb{BW}$. Again, using Proposition (V.1.1), we see that the submonoid $\mathcal{B}_S$ acts trivially (this is the easy direction of (V.1.1)) and so we obtain an action of $\mathrm{Doub}_S(C)$ on $\mathbb{BW}$ as defined above.

Finally we must show that the action is free. Suppose $\{M,f\}\cdot\{\mathbb{L},A,B\} = \{\mathbb{L},A,B\}$, so that $\{\mathbb{L}\otimes M, Af^2 Bf^3\} = \{\mathbb{L},A,B\}$. Using Proposition (V.1.1), we see that there are line bundles N_i on C and nonzero sections g_i of N_i such that $(\mathbb{L}\otimes M\otimes N_1, Af^2g_1^4, Bf^3g_1^6) \cong (\mathbb{L}\otimes N_2, Ag_2^4, Bg_2^6)$. Hence there is an isomorphism $\alpha:\mathbb{L}\otimes N_2 \to \mathbb{L}\otimes M\otimes N_1$ such that α^4 carries Ag_2^4 to $Af^2g_1^4$ and α^6 carries Bg_2^6 to Bfg_1^6. by tensoring α with the identity of $\mathbb{L}^{-1}\otimes N_1^{-1}$, we obtain an isomorphism $\beta:N_2\otimes N_1^{-1} \to M$ such that β^4 carries g_2^4/g_1^4 to f^2 and β^6 carries g_2^6/g_1^6 to f^3 (a priori as meromorphic sections, but in fact this shows that g_2^6/g_1^6 is holomorphic). Hence β^2 must carry $(g_2/g_1)^2$ to f, and since g_2/g_1 is a holomorphic section of $N_2\otimes N_1^{-1}$, we see that $[M,f] = [N_2\otimes N_1^{-1},(g_2/g_1)^2]$ is an element of $\mathcal{B}_S$. Hence $\{M,f\}$ is trivial in $\mathrm{Doub}_S(C)$. ■

Note that the J-map for an element of $\mathbb{BW}$ is well-defined; with this in mind, Proposition (V.2.1) can now be rephrased as follows.

(V.3.6)<u>Proposition:</u> Let $\{\mathbb{L},A,B\}$ be an element of $\mathbb{BW}$ whose J-map is not identically 0 or 1. Then the set of elements of $\mathbb{BW}$ with the same J-map as $\{\mathbb{L},A,B\}$ is exactly the orbit of $\{\mathbb{L},A,B\}$ under the action of the full double cover group Doub(C).

Hence the set of smooth minimal elliptic surfaces with section, all having the same J-map, is a torsor under the group Doub(C). It is natural to ask, when can a map $J:C \longrightarrow \mathbb{P}^1$ be a J-map for an elliptic surface with section? The answer is that there is no restriction.

(V.3.7)<u>Lemma:</u> Fix a non-constant map $J:C \longrightarrow \mathbb{P}^1$. Let J be defined by two sections [t:s] of $\mathbb{L} = J^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then the Weierstrass data $(\mathbb{L},-3t(t-s)s^2,2t(t-s)^2s^3)$ has J as its J-map.

Proof: This is just a calculation:

$$J(\mathbb{L},-3t(t-s)s^2,2t(t-s)^2s^3) = J(-3t(t-s)s^2,2t(t-s)^2s^3)$$
$$= [4(-3t(t-s)s^2)^3:4(-3t(t-s)s^2)^3+27(2t(t-s)^2s^3)^2]$$
$$= [-108t^3(t-s)^3s^6:-108t^3(t-s)^3s^6+108t^2(t-s)^4s^6]$$
$$= [t:t-(t-s)] \quad \text{(factoring out } -108t^2(t-s)^3s^6)$$
$$= [t:s] = J. \quad ■$$

The above data $(\mathbb{L},-3t(t-s)s^2,2t(t-s)^2s^3)$ is the pullback of the data $(\mathcal{O}(1),-3t(t-s)s^2,2t(t-s)^2s^3)$ on $\mathbb{P}^1$, where now [t:s] are the homogeneous coordinates. This data defines a rational elliptic surface with section over

$\mathbb{P}^1$, and an application of the Table (IV.3.1) shows that there are only three singular fibers: over 0 we have a fiber of type II, over 1 a fiber of type III, and over ∞ a fiber of type I_1^*. This surface has the identity for its J-map.

Of course any constant J-map is easy to obtain, as a product surface, and if J is not identically 0 or 1, the same formula above can be used. Therefore we have:

(V.3.8)<u>Proposition:</u> Let $J:C \longrightarrow \mathbb{P}^1$ be any map, not identically 0 or 1. Then the set of smooth minimal elliptic surfaces with section having this J as its J-map is a torsor under Doub(C).

In fact we have something a little more precise, using the special surface given above and Lemma (V.3.3).

(V.3.9)<u>Proposition:</u> Let $\pi:X \longrightarrow C$ be a smooth minimal elliptic surface with minimal Weierstrass data $(\mathbb{L},A,B)$. Let $J = [t,s]$ be the J-map for π, and suppose that J is not identically 0 or 1. Then there is a double cover pair (M,f) on C with $(f)_0$ reduced, unique up to isomorphism, such that $(\mathbb{L},A,B)$ is the minimilization of the twist of $(J^*\mathcal{O}_{\mathbb{P}^1}(1),-3t(t-s)s^2,2t(t-s)^2s^3)$ by (M,f). I.e., there is a unique nonnegative divisor D on C, and a section g of $\mathcal{O}_C(D)$ with zero locus D, such that

$$(\mathbb{L}(D),Ag^4Bg^6) \cong (J^*\mathcal{O}_{\mathbb{P}^1}(1)\otimes M,-3t(t-s)s^2f^2,2t(t-s)^2s^3f^3).$$

4: The transfer of * process

Suppose that C is a disk with center 0; then Pic(C) is trivial, and double cover pairs relative to {0} are all isomorphic to $(\mathcal{O}_C,t^n)$ for some $n \geq 0$; moreover, $\mathrm{Doub}_{\{0\}}(C)$ is a group with only two elements, represented by the identity $(\mathcal{O}_C,1)$ and the nontrivial element $(\mathcal{O}_C,t)$. It is the effect of performing a quadratic twist by this nontrivial element which I want to analyze briefly.

Let $(\mathcal{O}_C,A(t),B(t))$ be Weierstrass data over the disk C. Upon twisting by $(\mathcal{O}_C,t)$ we arrive at the data $(\mathcal{O}_C,t^2A(t),t^3B(t))$. If one uses the (a,b,δ) Table (IV.3.1) to determine the type of singular fiber over 0, one sees that (a,b,δ) is replaced by $(a+2,b+3,\delta+6)$ upon performing the quadratic twist. Since minimalizing is essentially subtracting $(4,6,12)$ from (a,b,δ), we recover again the statement that performing a quadratic twist twice has no effect after minimalizing.

Inspection of the Table (IV.3.1) gives the following table, for deciding what the fiber type over 0 is after the quadratic twist is performed.

(V.4.1)Table of effect of quadratic twist, locally

The fiber types are switched according to the rule:

$$I_N \longleftrightarrow I_N^* \text{ for any } N \geq 0$$
$$II \longleftrightarrow IV^*$$
$$III \longleftrightarrow III^*$$
$$IV \longleftrightarrow II^*$$

This is the so-called "transfer of *" on the fiber over 0. The "*-fibers", namely the fibers of types I_N^*, IV^*, III^*, and II^*, are switched with the "non-*-fibers" by the performing of a quadratic twist.

Note that a fiber is a *-fiber if and only if $(a,b,\delta) \geq (2,3,6)$, where (a,b,δ) is the triple of orders of vanishing of the minimal Weierstrass data.

Now let us suppose that C is a complete curve. Then if $(\mathbb{L},A,B)$ is Weierstrass data over C, and (M,f) is a double cover pair over C with $(f)_0$ reduced, the minimalization of the quadratic twist performs the transfer of * process at each fiber over the points of C where f is zero. Since $(f)_0$ is a divisor of even degree on C, we see that this transfer of * process occurs an even number of times, so that the parity of the number of *-fibers remains constant. We will say that Weierstrass data is *-even or *-odd depending on whether the minimalization has an even or odd number of *-fibers. By the above remarks, this is a property of the J-map for the Weierstrass data.

Hence, using appropriate double cover pairs, we may twist any Weierstrass fibration to one with either 0 or 1 *-fiber. Such a Weierstrass fibration (or Weierstrass data, or smooth minimal elliptic surface with section) will be said to be *-minimal.

There is a related notion for the J-map, as we now discuss. Assume that C is a complete curve. Weierstrass data $(\mathbb{L},A,B)$ is said to be J-minimal if $\deg(\mathbb{L})$ is minimal among all Weierstrass data having the same associated J-map.

(V.4.2)Lemma: Let $(\mathbb{L},A,B)$ be minimal Weierstrass data over a complete curve C. Assume that the associated J-map is not identically 0 or 1. Then $(\mathbb{L},A,B)$ is *-minimal if and only if it is J-minimal.

Proof: Assume first that $(\mathbb{L},A,B)$ is not *-minimal. Then there are at least two *-fibers, say over the points c_1 and c_2 of C. Let M be a line bundle on C such that $M^2 \cong \mathcal{O}_C(c_1+c_2)$, and let f be a section of $\mathcal{O}_C(c_1+c_2)$ such that $(f)_0 = c_1+c_2$. Then $(\mathbb{L}\otimes M^{-1},A/f^2,B/f^3)$ is Weierstrass data over C, with the same J-map as $(\mathbb{L},A,B)$, and since $\deg(M) = 1$, this has strictly smaller degree for the line bundle; hence $(\mathbb{L},A,B)$ is not J-minimal.

Conversely assume that $(\mathbb{L},A,B)$ is *-minimal, and let (M,f) be a double cover pair on C with $(f)_0$ reduced. We must show that $\deg(\mathbb{L}) \leq \deg(\mathbb{L}\otimes M(-D))$, where D is the divisor used to minimalize the quadratic twist $(\mathbb{L}\otimes M,Af^2,Bf^3)$. If $(f)_0$ has no points in common with the zeroes of the original discriminant $\Delta = 4A^3+27B^2$, then no minimalization is necessary and the result follows from the fact that $\deg(M) \geq 0$. In fact, if all points of $(f)_0$ are non-*-fibers, the same argument works: still no minimalization is required. Since we are assuming that $(\mathbb{L},A,B)$ is *-minimal, we are left with the case when there is exactly one *-fiber over a point c of C and c is a zero of f.

In this case the minimalization necessary is to twist by -c, i.e., in the notation above, $D = c$; since f has zeroes, $\deg(M) \geq 1$, so that in this case $\deg(M(-c)) \geq 0$ and the result is proven. ■

Lecture VI: The J-map and Monodromy.

1: Uniqueness of germs of fibers

Let $\pi : X \longrightarrow C$ be an elliptic surface with section, and let c be a point of C. The germ of the fiber $\pi^{-1}(c)$ of π over c is what you expect: the equivalence class of the fibration restricted to a neighborhood of c, two fibrations being equivalent as germs if they are isomorphic after shrinking the neighborhoods. It is our task in this section to demonstrate that the germ of a fiber is determined by very little data.

We will say that the multiplicity of a constant function is ∞: if f is constant near c, then $\mathrm{mult}_c(f) = \infty$.

(VI.1.1)Proposition: The germ of a fiber $\pi^{-1}(c)$ of an elliptic surface with section $\pi : X \longrightarrow C$ is determined by J(c), $\mathrm{mult}_c(J)$, and the singular fiber type.

Proof: We may assume that C is a small disk around c = 0, and that the only singular fiber of π is over 0. Assume first that J is not identically 0 or 1. Then J is determined, locally analytically, by J(0) and $\mathrm{mult}_0(J)$. In this case by Proposition (V.3.8), the possibilities for π form a torsor under Doub(C). However if we insist that there are no singular fibers away from 0, we have that the possibilities form a torsor under the subgroup $\mathrm{Doub}_{\{0\}}(C)$, which as described in (V, section 4), is a group of order two. The singular fiber distinguishes between the two possibilities: one is a "*-fiber", the other is not.

If J is identically 0, then locally π has a Weierstrass equation of the form $y^2 = x^3 + t^b$ for some $b \geq 0$, and minimality forces $b \leq 5$. These give 6 different singular fiber types, by Table (IV.3.1). If J is identically 1, then locally π has a Weierstrass equation of the form $y^2 = x^3 + t^a x$, with $0 \leq a \leq 3$; these give 4 different singular fibers. Hence in either case the singular fiber determines the exponent, which determines the germ. ∎

In fact, the same statement is true without assuming the existence of a section, but only assuming that there are no multiple fibers; this follows since there is a local section in that case.

In the following table we present local normal forms for germs of fibers, in terms of the Weierstrass coefficients (A,B).

(VI.1.2) Table of normal forms for Weierstrass fibrations over the t-disk

Fiber	J(t)	(A,B)
I_0	0	$(0,1)$
	1	$(1,0)$
	$j\neq 0,1$	$(-3j(j-1),2j(j-1)^2)$
	t^{3n}	$(t^n,1)$
	$1+t^{2n}$	$(1,t^n)$
	$j+t^n,\ j\neq 0,1$	$(-3(j+t^n)(j+t^n-1),2(j+t^n)(j+t^n-1)^2)$
I_N	t^{-N}	$(-3(1-t^N),2(1-t^N)^2)$
I_0^*	0	$(0,t^3)$
	1	$(t^2,0)$
	$j\neq 0,1$	$(-3j(j-1)t^2,2j(j-1)^2t^3)$
	t^{3n}	(t^{n+2},t^3)
	$1+t^{2n}$	(t^2,t^{n+3})
	$j+t^n,\ j\neq 0,1$	$(-3t^2(j+t^n)(j+t^n-1),2t^3(j+t^n)(j+t^n-1)^2)$
I_N^*	t^{-N}	$(-3t^2(1-t^N),2t^3(1-t^N)^2)$

Fiber	J(t)	(A,B)	Fiber	J(t)	(A,B)
II	0	$(0,t)$	IV^*	0	$(0,t^4)$
	t^{3n+1}	(t^{n+1},t)		t^{3n+1}	(t^{n+3},t^4)
III	1	$(t,0)$	III^*	1	$(t^3,0)$
	$1+t^{2n+1}$	(t,t^{n+2})		$1+t^{2n+1}$	(t^3,t^{n+5})
IV	0	$(0,t^2)$	II^*	0	$(0,t^5)$
	t^{3n+2}	(t^{n+2},t^2)		t^{3n+2}	(t^{n+4},t^5)

2: Local monodromy

Let $\pi : X \longrightarrow C$ be an elliptic surface over a small disk C, such that theonly singular fiber is over 0. If one fixes $c \neq 0$, one has the smooth fiber $X_c = \pi^{-1}(c)$, and its integral first homology $H^1(X_c,\mathbb{Z})$, which is a free abelian group of rank 2. Choose a loop γ on C which winds once around the origin, in the positive (counterclockwise) direction, starting and ending at c. The elements of $H^1(X_t,\mathbb{Z})$ move continuously as t moves along the loop γ, giving in effect an identification of $H^1(X_t,\mathbb{Z})$ with $H^1(X_c,\mathbb{Z})$. Upon returning to c via the loop γ, we have an automorphism of $H^1(X_c,\mathbb{Z})$. This automorphism is independent of the choice of γ (it depends only on the homotopy class of γ). We thus obtain a well-defined element of $\mathrm{Aut}_{\mathbb{Z}}(H^1(X_c,\mathbb{Z}))$, called the <u>local monodromy around 0</u>. It is in fact independent of c, for c chosen sufficiently close to 0.

Choose a nowhere zero holomorphic 1-form ω on X_c. Then integration of ω along the elements of $H^1(X_c,\mathbb{Z})$ gives an isomorphism of $H^1(X_c,\mathbb{Z})$ with the lattice of periods Λ_c. We may thus view the local monodromy as an automorphism of this lattice. Since $X_c \cong \mathbb{C}/\Lambda_c$, this is a useful remark for computations. In particular, given a basis $\{\tau_1,\tau_2\}$ for Λ_c, the orientation of the τ_i's is preserved upon continuation around the loop γ (the continuation occurs inside $H^1(X_t,\mathbb{C})$, which is isomorphic to $\mathbb{C}$, and the complex structure is preserved); therefore the local monodromy must have determinant 1.

If one fixes a basis of $H^1(X_c,\mathbb{Z})$, then one may regard the local monodromy as an element of $SL(2,\mathbb{Z})$; if one does not fix a basis, then one has naturally a conjugacy class of elements in $SL(2,\mathbb{Z})$. One is usually a bit sloppy with language and refers to all of these manifestations of the monodromy around 0 as the local monodromy.

Of course, the local monodromy only depends on the germ of the fiber, and so one can calculate the local monodromy for the normal forms only, to obtain the local monodromy in all cases. In fact, the local monodromy does not depend on the local normal form, but only on the type of singular fiber. We present in the table below the results.

(VI.2.1)<u>Table</u> of representatives in $SL(2,\mathbb{Z})$ for the local monodromy.

<u>Fiber</u>	<u>Local monodromy</u>	<u>Fiber</u>	<u>Local monodromy</u>
I_N	$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$	I_N^*	$\begin{pmatrix} -1 & -N \\ 0 & -1 \end{pmatrix}$
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Since the local monodromy distinguishes between the singular fiber types, we get the following from (VI.1.1).

(VI.2.2)<u>Corollary</u> The germ of a fiber $\pi^{-1}(c)$ of an elliptic surface with section $\pi:X \longrightarrow C$ is determined by $J(c)$, $\mathrm{mult}_c(J)$, and the local monodromy.

Some remarks about the calculation of the local monodromy will be postponed until after a discussion of local base change is made.

3: Global monodromy: the homological invariant

The j-map from the upper half-plane $\mathfrak{H}$ to $\mathbb{C}$, defined by $j(\tau) = J(\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau))$, (actually J is defined from j, not the other way around) exhibits $\mathfrak{H}$ as a branched cover of $\mathbb{C}$, branched only over the two points $\{0,1\}$. The unramified covering $j:\mathfrak{H} - j^{-1}\{0,1\} \longrightarrow \mathbb{C} - \{0,1\}$ is a $PSL(2,\mathbb{Z})$-cover, and therefore gives naturally a map $\alpha:\pi_1(\mathbb{C} - \{0,1\}) \longrightarrow PSL(2,\mathbb{Z})$.

Suppose that $J:C \longrightarrow \mathbb{P}^1$ is any nonconstant map, and assume that $S \subset C$ is a finite set containing $J^{-1}(\{0,1,\infty\})$; i.e., on $C - S$, J is never 0, 1, or ∞. Then $J:C - S \longrightarrow \mathbb{C} - \{0,1\}$ is well-defined and not constant, and so gives a map $J_*:\pi_1(C - S) \longrightarrow \pi_1(\mathbb{C} - \{0,1\})$; after composing with the natural map α above, we obtain a map (which we will also call J_*) $J_*:\pi_1(C - S) \longrightarrow PSL(2,\mathbb{Z})$.

Suppose now that J is the J-map for an elliptic surface over C, and that C is a small disk about 0, as in the previous section. We see that J_* can be thought of as the monodromy of the period τ, after making the normalization of the lattice Λ_c with $\mathbb{Z}+\mathbb{Z}\tau$ before and after going around the loop. Of course to remain in the upper half-plane we must replace τ by $-\tau$ if necessary; this is part of the normalization which may be required after going around the loop. Hence we see that J_* is slightly cruder than the local monodromy: it doesn't distinguish between $\pm\tau$. In any case we certainly have a commutative diagram, locally:

$$\begin{array}{ccc} \pi_1(C - 0) & \xrightarrow{\text{local monodromy}} & SL(2,\mathbb{Z}) \\ & \searrow^{J_*} & \downarrow \\ & & PSL(2,\mathbb{Z}) \end{array}$$

where the vertical arrow is the natural quotient map, modding out by $\pm ID$. The commutativity is "up to conjugacy", since the local monodromy is defined up to conjugacy.

It is clear that this globalizes, in case C is not simply a disk. If $S \subset C$ is a finite set containing the discriminant locus (so that the fibers over the points of $C - S$ are all smooth), we have the <u>global monodromy</u> $G:\pi_1(C - S) \longrightarrow SL(2,\mathbb{Z})$ (after choosing a base point c and a basis for $H^1(X_c,\mathbb{Z})$). The choice of c is inconsequential. This global monodromy is called the <u>homological invariant</u> of the elliptic surface over C.

Given any finite set S of C, and a representation G of $\pi_1(C-S)$ into $SL(2,\mathbb{Z})$, and a nonconstant map $J:C \longrightarrow \mathbb{P}^1$ such that $J^{-1}\{0,1,\infty\} \subseteq S$, we say that G belongs to J if the diagram above commutes, up to conjugacy. This language is due to Kodaira. The discussion above shows the following.

(VI.3.1)Proposition: Let $\pi:X \longrightarrow C$ be an elliptic surface with no multiple fibers, and let $S \subset C$ be a finite subset such that $J^{-1}\{0,1,\infty\} \subseteq S$ and π is smooth outside of S. Then the homological invariant G for π belongs to J.

Therefore, in the elliptic surface situation, G is exactly a lift of J_* from $PSL(2,\mathbb{Z})$ to $SL(2,\mathbb{Z})$. If S has $|S|$ points, then $\pi_1(C-S)$ is generated by $2g+|S|$ elements, subject to one relation; and this relation may be used to solve for the one of the generators. Since there are two possibilities for a lift of a generator from PSL to SL, we see the following.

(VI.3.2)Lemma: Given a non-constant map $J:C \longrightarrow \mathbb{P}^1$, and a finite set $S \subset C$ such that $J^{-1}\{0,1,\infty\} \subseteq S$, the number of homological invariants G belonging to J is $2^{2g+|S|-1}$.

The reader should compare this with Lemma (V.3.4): this number is exactly the same as the order of the relevant double cover group. Hence one suspects that there should be a 1-1 correspondence between these lifts and elements of $Doub_S(C)$. The elements of $Doub_S(C)$ are in 1-1 correspondence with the elliptic surfaces with section, with singular fibers lying over S, and so we suspect that there should be a 1-1 correspondence between homological invariants (belonging to a fixed J) and elliptic surfaces with section, with that J. This is indeed the case.

Note that the representation G, or indeed its conjugacy class, gives a locally constant sheaf (locally isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$) over C-S; in fact the data of the conjugacy class of G is equivalent to the data of an isomorphism class of such a sheaf. We will refer to this sheaf as G, also, and also call it the homological invariant.

(VI.3.3)Proposition: Given a non-constant map $J:C \longrightarrow \mathbb{P}^1$ and a representation $G:\pi_1(C-S) \longrightarrow SL(2,\mathbb{Z})$ belonging to J, there exist a unique Weierstrass fibration X(J,G) with J-map J and homological invariant G.

Proof: The local existence of X(J,G) follows from Lemma (V.3.7) and the fact that all possible local monodromy is realizable. The local uniqueness is Corollary (VI.1.3). So we must only address global considerations.

Choose a covering $\{\mathcal{U}_i\}$ for C by sufficiently small analytic disks, such that each s in S is contained in exactly one of the $\mathcal{U}_i$'s; moreover, choose them so that $\mathcal{U}_i \cap \mathcal{U}_j$ is also a disk for every i and j. Let $\pi_i : X_i \to \mathcal{U}_i$ be the unique elliptic surface with section σ_i over $\mathcal{U}_i$ with J-map equal to $J|_{\mathcal{U}_i}$ and local monodromy given by the sheaf (with fibers $\mathbb{Z}\oplus\mathbb{Z}$) G. Thus we have fixed an identification of $R^1\pi_{i*}\mathbb{Z}$ (the sheaf of local $H^1(X_t,\mathbb{Z})$'s) with $G|_{\mathcal{U}_i}$; call this identification α_i.

Over $\mathcal{U}_i \cap \mathcal{U}_j$, the fibrations π_i and π_j have the same J-map, and trivial monodromy, so there is an isomorphism $\phi_{ij}:X_j \longrightarrow X_i$ respecting the fibrations π_i and π_j, mapping the section σ_j to σ_i, and finally respecting the identifications made on the sheaf of local $H^1(X_t,\mathbb{Z})$'s: $\alpha_i \circ \phi_{ij*} = \alpha_j$ on $\mathcal{U}_i \cap \mathcal{U}_j$. Moreover the isomorphism ϕ_{ij}, with these requirements, is unique: the only automorphisms possible which preserve the section all act nontrivially on the sheaf of H^1's.

For three indices i,j, and k, the composition $\phi_{ij}\circ\phi_{jk}\circ\phi_{ki}$ is an automorphism of $X_i|_{\mathcal{U}_i\cap\mathcal{U}_j\cap\mathcal{U}_k}$ which preserves the fibration, the section σ_i and acts as the identity on the sheaf of local $H^1(X_t,\mathbb{Z})$'s. Hence it is the identity, and the local isomorphisms ϕ_{ij} patch together to give an elliptic surface with section $\pi:X(J,G) \to C$.

The uniqueness of X(J,G) follows from the uniqueness of the local isomorphisms ϕ_{ij} in the construction above. ■

(VI.3.4)<u>Proposition:</u> Given a non-constant map $J:C \to \mathbb{P}^1$ and a representation $G:\pi_1(C-S) \longrightarrow SL(2,\mathbb{Z})$ belonging to J, the set of elliptic surfaces without multiple fibers, with that given J and homological invariant G, is in 1-1 correspondence with the set

$$H^1(C,\text{sheaf of local sections of } X(J,G)).$$

4: Base changes of fibers

Let $\pi:X \to C$ be an elliptic surface with section over the t-disk. After making a base change of order M (replacing t by s^M), we obtain the M^{th}-order base change of the family. Clearly the germ after the base change depends only on M and the germ before making the base change, so by uniqueness, using the normal forms given above, we can readily determine the fiber type of the singular fiber after the base change: it depends only on the fiber type before the base change. We present the information below.

(VI.4.1)Table of fibers after a base change of order M.

Before	M	After
I_0	$M\geq 1$	I_0
I_N	$M\geq 1$	I_{MN}
I_N^*	0 mod 2	I_{MN}
	1 mod 2	I_{MN}^*
II	0 mod 6	I_0
	1 mod 6	II
	2 mod 6	IV
	3 mod 6	I_0^*
	4 mod 6	IV^*
	5 mod 6	II^*
III	0 mod 4	I_0
	1 mod 4	III
	2 mod 4	I_0^*
	3 mod 4	III^*
IV	0 mod 3	I_0
	1 mod 3	IV
	2 mod 3	IV^*

Before	M	After
II^*	0 mod6	I_0
	1 mod6	II^*
	2 mod6	IV^*
	3 mod6	I_0^*
	4 mod6	IV
	5 mod6	II
III^*	0 mod 4	I_0
	1 mod 4	III^*
	2 mod 4	I_0^*
	3 mod 4	III
IV^*	0 mod 3	I_0
	1 mod 3	IV^*
	2 mod 3	IV

In particular, note that the fibers I_0^*, II, III, IV, IV^*, III^*, and II^* all may be base-changed to a smooth fibration, after base changes to order 2, 6, 4, 3, 3, 4, and 6, respectively. Conversely, the germs for these singular fibers may be constructed as quotients of smooth germs, by cyclic groups with these orders. After doing so, the monodromy is rather easy to determine, and we leave it to the reader to verify these parts of the monodromy table.

For the fibers of type I_N^*, a base change to order 2 will give type I_{2N}, and so understanding the stable case is enough information.

The monodromy for I_1 is $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, and it is not hard to verify that one can assume that the period $\tau(t)$ for the lattice is $\tau(t) = \frac{1}{2\pi i}\log(t)$. For I_N, which is a base change of order N of I_1, we have $\tau(t) = \frac{1}{2\pi i}\log(t^N) = \frac{N}{2\pi i}\log(t)$.

Hence we may construct this I_1 germ as $\mathbb{C} \times \Delta/\mathbb{Z}+\mathbb{Z}\frac{1}{2\pi i}\log(t)$ over the disk Δ. By using the exponential $e^{2\pi i-}$ to factor out the $\mathbb{Z}$ first, we see that we can put the family into the form $\mathbb{C}^* \times \Delta/\{t^j|j\in\mathbb{Z}\}$. This is the "Jacobi form" of the local family. For a fiber of type I_N, the Jacobi form is $\mathbb{C}^* \times \Delta/\{t^{Nj}|j\in\mathbb{Z}\}$.

The I_N^* fibers can be obtained, as noted above, by taking the quotient of an I_{2N} germ by a $\mathbb{Z}/2\mathbb{Z}$-action; this action, as the reader can check, is given locally by $(z,t) \longmapsto (-z,-t)$. The local form for the period lattice is $\mathbb{Z}t^{1/2}+\mathbb{Z}t^{1/2}\log(t)/2\pi i$ if $N \geq 1$. Note that the periods themselves are multi-valued (that's the monodromy!) but the lattice is well-defined for each fixed t.

The normal forms for the other lattices are given in the table below.

(VI.4.2)<u>Table</u> of local normal forms over the t-disk C for the lattice Λ_t of periods. We write Λ_t as $\mathbb{Z}\tau_1(t) + \mathbb{Z}\tau_2(t)$.

<u>Fiber</u>	<u>J(t)</u>	$\tau_1(t)$	$\tau_2(t)$
I_0	any	1	$\tau(t)$, $\tau(t)$ holomorphic on C
I_0^*	any	$t^{1/2}$	$t^{1/2}\tau(t)$, $\tau(t)$ holomorphic on C
$I_{N\geq 1}$	t^{-N}	1	$\frac{N}{2\pi i}\log(t)$
$I_{N\geq 1}^*$	t^{-N}	$t^{1/2}$	$t^{1/2}\frac{N}{2\pi i}\log(t)$
II	0	$t^{5/6}$	$t^{5/6}e^{2\pi i/3}$
	t^{3n+1}	$t^{5/6}(1-t^{n+1/3})$	$t^{5/6}(e^{2\pi i/3} - e^{4\pi i/3}t^{n+1/3})$
III	1	$t^{3/4}$	$t^{3/4}i$
	$1+t^{2n+1}$	$t^{3/4}(1-t^{n+1/2})$	$t^{3/4}i(1 + t^{n+1/2})$
IV	0	$t^{2/3}$	$t^{2/3}e^{2\pi i/3}$
	t^{3n+2}	$t^{2/3}(1-t^{n+2/3})$	$t^{2/3}(e^{2\pi i/3} - e^{4\pi i/3}t^{n+2/3})$
IV^*	0	$t^{1/3}$	$t^{1/3}e^{2\pi i/3}$
	t^{3n+1}	$t^{1/3}(1-t^{n+1/3})$	$t^{1/3}(e^{2\pi i/3} - e^{4\pi i/3}t^{n+1/3})$
III^*	1	$t^{1/4}$	$t^{1/4}i$
	$1+t^{2n+1}$	$t^{1/4}(1-t^{n+1/2})$	$t^{1/4}i(1 + t^{n+1/2})$
II^*	0	$t^{1/6}$	$t^{1/6}e^{2\pi i/3}$
	t^{3n+2}	$t^{1/6}(1-t^{n+2/3})$	$t^{1/6}(e^{2\pi i/3} - e^{4\pi i/3}t^{n+2/3})$

The reader should beware that the matrices for the monodromy are computed using τ_2 as the first basis vector and τ_1 as the second! This is traditional, because the action of monodromy on the upper-half plane $\mathfrak{H}$ is usually given by sending τ to $\frac{a\tau+b}{c\tau+d}$, with associated matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; this places τ first and 1 second in the order for the generators of the lattice. We have usually written the lattice, however, in the form $\mathbb{Z} + \mathbb{Z}\tau$, which places 1 first and τ second.

Lecture VII: The Neron-Severi Group and the Mordell-Weil Group

1: The Picard group and the Neron-Severi Group

Let $\pi:X \longrightarrow C$ be a smooth minimal elliptic surface.

(VII.1.1)Lemma:
(a) $\pi^*:\mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(X)$ is injective.
(b) If π has a section and X is not a product, then $\pi^*:\mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^0(X)$ is an isomorphism.

Proof: Statement (a) follows from the projection formula, using that $\pi_*\mathcal{O}_X \cong \mathcal{O}_C$: for any line bundle $\mathcal{L}$ on C, $\pi_*\pi^*\mathcal{L} \cong \mathcal{L}\otimes\pi_*\mathcal{O}_X \cong \mathcal{L}$. To see (b), note that since X is not a section, the irregularity q equals the genus g of C; therefore both $\mathrm{Pic}^0(C)$ and $\mathrm{Pic}^0(X)$ are both tori of the same dimension, and since π^* is injective by part (a), it must be an isomorphism on the Pic^0's. ■

(VII.1.2)Lemma: Assume π has a section and the associated line bundle $\mathbb{L}$ has positive degree. Then $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is torsion-free, i.e., there are no torsion classes in Pic(X) not algebraically equivalent to 0; from the previous lemma, we have then that every torsion classs comes from a torsion class on C.

Proof: Let T be a divisor representing a torsion class, which we assume to be nonzero. Then $H^0(X,T) = 0$, so by Riemann-Roch we have
$h^2(X,T) \geq T(K-T)/2 + \chi = \chi = \deg(\mathbb{L}) \geq 1$. Hence there is an effective divisor D in $|K_X - T|$. Recall that K is pulled back from C, so that $D\cdot F = 0$; hence D is vertical. However since $D^2 = 0$, we must have that D is a sum of complete fibers: the vertical divisors have negative semi-definite intersection form, with only complete fibers having square zero. However K, being pulled back from C, can be written as a sum of complete fibers; therefore so can T, and so T is pulled back from C. Hence the class of T is zero mod $\mathrm{Pic}^0(C)$, which by the previous lemma means that the class of T is zero mod $\mathrm{Pic}^0(X)$ too. ■

(VII.1.3)Corollary: Assume that π has a section and $\deg(\mathbb{L}) \geq 1$. Then Neron-Severi group NS(X) is isomorphic to $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. Moreover the quotient Pic(X)/Pic(C) is isomorphic to $NS(X)/\mathbb{Z}\overline{F}$, where $\overline{F}$ is the class of a fiber of π.

Proof: The first statement is simply that there is no torsion in $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. The second follows from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(C) & \longrightarrow & \mathrm{Pic}(C) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow \pi^* & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{NS}(X) & \longrightarrow & 0 \end{array}$$

after noting that the image of $\pi^*\mathbb{Z}$ is $\mathbb{Z}\overline{F}$ in NS(X). ■

(VII.1.4)<u>Lemma:</u> Let X_η be the general fiber of π.

(a) Any divisor D on X can be written uniquely as D = V + H with V vertical and H horizontal.

(b) The restriction map $r:\mathrm{Div}(X) \longrightarrow \mathrm{Div}(X_\eta)$ is surjective with the set of vertical divisors as kernel. We denote r(D) by D_η.

(c) $r:\mathrm{PrinDiv}(X) \longrightarrow \mathrm{PrinDiv}(X_\eta)$ is surjective.

(d) $r:\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_\eta)$ is surjective with $\{\mathcal{O}_X(V) \mid V \text{ is vertical}\}$ as kernel.

(e) If D is a divisor on X with D_η linearly equivalent to 0 in X_η, then D is linearly equivalent to a vertical divisor on X.

Proof: Statement (a) is obvious. To see the surjectivity of r on Div, one simply takes the closure $\overline{E}$ of a divisor E in $\mathrm{Div}(X_\eta)$: then $(\overline{E})_\eta = E$. Clearly the vertical divisors are in the kernel of r, and no horizontal divisor is, proving (b). Statement (c) follows since the function field of X_η is the same as that of X: $K(X_\eta) = K(X)$. Statement (e) follows from writing Pic as Div/PrinDiv and using (b) and (c). Finally, the last statement is a consequence of (e). ■

(VII.1.5)<u>Corollary</u> Suppose π has a section and $\deg(\mathbb{L}) \geq 1$. Then there is a well-defined map $r:\mathrm{NS}(X) \longrightarrow \mathrm{Pic}(X_\eta)$ induced by restriction of divisors. Moreover this map is surjective with kernel generated by the classes of vertical divisors.

Proof: Clearly r is 0 on $\pi^*\mathrm{Pic}(C)$, so the existence of r follows from (VII.1.3), and the surjectivity from (d) above; the statement about the kernel also follows from (d). ■

2: The Mordell-Weil group of sections and the Shioda-Tate formula

We assume in this section that $\pi:X \longrightarrow C$ is an elliptic surface with a given section S_0, and associated bundle $\mathbb{L}$.

Let MW(X) be the set of sections of π; addition, fiber by fiber, induces a group law on MW(X) with S_0 as the zero. This group is called the Mordell-Weil Group of π (or of X).

The group law can also be described as follows. Note that MW(X) can be identified with the set of rational points on the general fiber X_η: a section gives a point by restriction, and a point gives a section as its closure. The point of X_η corresponding to the zero section S_0 will be denoted by p_0. The sum in MW(X) is then inherited from the sum on the points of X_η, which is after all an elliptic curve over K(C) and as such its points form an abelian group. More explicitly, let S_1 and S_2 be two sections. Then the sum $S_1 \oplus S_2$ in MW(X) is the section S_3, where $(S_1+S_2-S_0)_\eta \sim (S_3)_\eta$.

For any divisor E on X_η, one has the summation $\sum E$, defined by adding in the group law on X_η the points of E. This gives a homomorphism $\sum:\mathrm{Div}(X_\eta) \longrightarrow \mathrm{MW}(X)$, which by Abel's theorem on X_η factors through $\mathrm{Pic}(X_\eta)$. Composition with the map $r:\mathrm{NS}(X) \longrightarrow \mathrm{Pic}(X_\eta)$ gives a homomorphism β from NS(X) to MW(X): $\beta(\bar{D}) = \overline{\sum(D_\eta)}$ = the class of the closure of $\sum(D_\eta)$.

(VII.2.1)Theorem: Let $A \subset \mathrm{NS}(X)$ be the subgroup generated by the class of the zero section S_0 and the vertical classes. Then the sequence

$$0 \longrightarrow A \xrightarrow{\alpha} \mathrm{NS}(X) \xrightarrow{\beta} \mathrm{MW}(X) \longrightarrow 0$$

is exact, where α is the inclusion.

Proof: Clearly β is surjective: If S is any section in MW(X), the class of S in NS(X) goes to S under β. Of course α is injective, and since $\beta(S_0) = S_0$ which is the zero of MW(X), and $\beta(V) = 0$ for any vertical V, we have that $\beta\circ\alpha = 0$. Let $D \in \ker(\beta)$. Then $\sum(D_\eta) = p_0$; therefore $D_\eta - \deg(D_\eta)p_0$ is linearly equivalent to 0 on X_η by Abel's Theorem. Since $\deg(D_\eta) = (D\cdot F)$, we have that $D - (D\cdot F)S_0$ restricts to 0 on X_η. Therefore by Lemma (VII.1.4(e)), $D - (D\cdot F)S_0$ is linearly equivalent to a vertical divisor V; hence $D = (D\cdot F)S_0 + V$ as classes in NS(X), so $D \in A$. ∎

(VII.2.2)Corollary: MW(X) is a finitely generated abelian group.

Denote by R the sublattice of A generated by vertical components not meeting S_0. R is a direct sum of root lattices of types A_N, D_N, E_6, E_7, and E_8; the root lattice is given in Table (II.3.1). R is an even negative definite lattice, and

$$\mathrm{rk}(R) = \sum_{c\in\Delta}(\#\text{ of components of } X_c - 1),$$

where $\Delta \subset C$ is the discriminant locus. Note that this local number is called "r" in Table (IV.3.1); hence we have $\mathrm{rk}(R) = \sum r_c$.

Note that the sublattice U of A generated by S_0 and the fiber F is a rank two unimodular sublattice, with R as its perpendicular space. Therefore

(VII.2.3) $\quad A = \langle S_0, F\rangle \overset{\perp}{\oplus} R = U \overset{\perp}{\oplus} R.$

In particular, $\mathrm{rk}(A) = 2 + \mathrm{rk}(R)$. This gives the following corollary. Denote by ρ the rank of NS(X), the Picard number of X.

(VII.2.4)<u>Corollary</u> (The Shioda-Tate formula): $\quad \rho = 2 + \sum_{c\in\Delta} r_c + \mathrm{rk}(MW(X)).$

Since U is unimodular, it splits off NS(X) also, giving the exact sequence

(VII.2.5) $\quad 0 \longrightarrow R \longrightarrow U^{\perp} \longrightarrow MW(X) \longrightarrow 0$

where $U^{\perp}$ is the perpendicular space to U in NS(X); this version of Theorem (VII.2.1) is sometimes useful.

Let L be any free finitely generated $\mathbb{Z}$-module with a nondegenerate bilinear form $\langle -,-\rangle$ with values in $\mathbb{Z}$. The form extend to a $\mathbb{Q}$-valued form on $L_{\mathbb{Q}} = L\otimes\mathbb{Q}$. Denote by $L^{\#}$ the module $L^{\#} = \{x \in L_{\mathbb{Q}} \mid \langle x,\ell\rangle \in \mathbb{Z} \text{ for all } \ell \in L\}$. $L^{\#}$ is a free-$\mathbb{Z}$-module containing L as a submodule of finite index; the quotient group $G_L = L^{\#}/L$ is a finite abelian group whose order is the discriminant of L.

$L^{\#}$ may be naturally identified with the dual module $L^{*} = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$, by sending $x \in L^{\#}$ to the functional $\langle x,-\rangle$.

The intersection form on NS(X) gives a map $NS(X) \longrightarrow R^{*} = \mathrm{Hom}(R,\mathbb{Z})$, which after identifying R^{*} with $R^{\#}$ and passing to the quotient G_R of $R^{\#}$ by R, gives a map $\gamma: NS(X) \longrightarrow G_R$. Note that $\gamma(A) = 0$, since S_0 and F do not meet any components generating R, and R goes to 0 in G_R. Therefore γ factors through MW(X), and we have a map (which we also call γ) $\gamma: MW(X) \longrightarrow G_R$.

Because of the importance of G_R we would like some more detailed information about it. Note that if Δ' is the set of points c of C such that

X_c is reducible (hence X_c contributes to R), then R is the orthogonal direct sum of the "local R's", namely $R = \overset{\perp}{\oplus}_{c\in\Delta'} R_c$, where R_c is the lattice generated by components of X_c not meeting S_0. Therefore G_R also splits as $G_R = \overset{\perp}{\oplus}_{c\in\Delta'} G_{R_c}$. Hence the calculation of G_R is local, and can be done once and for all for each fiber type. We present the results below, which we urge the reader to check.

(VII.2.6)<u>Table</u> of G_{R_c}.

<u>Fiber</u>=X_c	G_{R_c}
$I_{N\geq 2}$	$\mathbb{Z}/N$
I_N^*	$\mathbb{Z}/2\times\mathbb{Z}/2$ if N is even; $\mathbb{Z}/4$ if N is odd
II, II^*	{0}
III, III^*	$\mathbb{Z}/2$
IV, IV^*	$\mathbb{Z}/3$

<u>Moreover</u> the nonzero elements of G_{R_c} are exactly the cosets of the duals of the multiplicity one components of X_c, i.e., the cosets mod R of the functionals $e^\#$ which take value 1 on a multiplicity one component e of R and 0 on all other components.

Note that the order of G_{R_c} is the number of multiplicity one components of X_c: this is the column "d" of Table (IV.3.1).

Let

$$MW_0(X) = \{S \in MW(X) \mid S \text{ and } S_0 \text{ meet the same component of } X_c \text{ for every c in C}\}.$$

(VII.2.7)<u>Proposition:</u> $MW_0(X) = \ker(\gamma)$.

Proof: Certainly $\gamma(MW_0(X)) = 0$; the sections in $MW_0(X)$ cannot meet any components generating R, by the definition of R. Suppose S is not in $MW_0(X)$; then there is a reducible fiber X_c such that S meets a component e of R_c, with e having multiplicity one in X_c. Therefore $\gamma(S)$ projects onto the coset of $e^\#$ in G_{R_c}, since S meets e once and meets no other generator of R_c. Using (VII.2.6), we have that this coset is one of the nonzero elements of G_{R_c}, hence $\gamma(S) \neq 0$. ■

(VII.2.8)<u>Corollary</u>: $MW_0(X)$ has finite index in $MW(X)$.

(VII.2.9)<u>Lemma:</u> $MW_0(X)$ is torsion-free if $\deg(\mathbb{L}) \geq 1$.

Proof: Suppose that S is a torsion section of order n in $MW_0(X)$. Hence $nS - nS_0$ restricts to 0 on the general fiber X_η; therefore $nS - nS_0 = V$ with V vertical. However since S is in $MW_0(X)$, $S - S_0$ does not meet any vertical components; therefore neither does V, and so V must have square 0, forcing V to be a sum of fibers. Working in NS(X), we then have $nS - nS_0 = aF$ for some integer a.

Let $k = S \cdot S_0$ and $\ell = -\deg(\mathbb{L}) = S^2 = S_0^2$. By intersecting the above equation with S we obtain $n\ell - nk = a$, and by intersecting it with S_0 we get $nk - n\ell = a$; therefore $a = 0$ and $k = \ell$. However k is non-negative and ℓ is negative!. ∎

3: **Torsion in MW**

Denote by TMW(X) the torsion subgroup of MW(X), i.e., the group of torsion sections of X. We have the following fact. From the previous Lemma, we have the following:

(VII.3.1)<u>Corollary</u>: Suppose that $\deg(\mathbb{L}) \geq 1$. Then $\gamma: TMW(X) \longrightarrow G_R$ is injective.

Note that in particular, if $\deg(\mathbb{L}) \geq 1$, then a torsion section is completely determined by which vertical components it meets.

(VII.3.2)<u>Proposition:</u> Let S_1 and S_2 be two torsion sections of X. Then S_1 and S_2 are disjoint.

Proof: It suffices to prove that if S is a torsion section, then S does not meet S_0. Moreover, note that since sections meet only multiplicity one components of fibers, if the statement is true after making a base change, then it is true before. Therefore it suffice to prove the statement locally around a semistable fiber of type I_N, with $N \geq 0$, using Table (VI.4.1): any fiber has a semistable fiber as some base change.

Assume then that S and S_0 meet at a point of a smooth fiber of type I_0.

We may locally represent the fibration as $\mathbb{C}\times\Delta_t/\mathbb{Z}+\mathbb{Z}\tau(t)$, where Δ_t is a disk with coordinate t. The zero section is given by the map $z(t) = 0$. Suppose that S is given by the holomorphic map $z(t)$; since S is a torsion section, $z(t) \in \mathbb{Q}+\mathbb{Q}\tau(t)$ for all t; this forces $z(t)$ to be constant. Since we assume that S and S_0 meet, we have $z(0) = 0$, so $z(t) = 0$ and $S = S_0$.

Assume finally that S and S_0 meet at a point of a singular fiber of type I_N, with $N \geq 1$. We may locally represent the fibration in Jacobi form $\mathbb{C}^* \times \Delta_t/\{t^{Nj} | j\in\mathbb{Z}\}$, with the section S_0 given by $z(t) \equiv 1$. Let S be given locally by $z(t) \in \mathbb{C}^*$, with $z(0) = 1$ (because S and S_0 meet). Since S is torsion, we have $z(t)^k = t^{Nj}$ for some integer j. Hence $z(t)$ is a branch of $t^{Nj/k}$, and since $z(t)$ is holomorphic, we have that $Nj/k \in \mathbb{Z}$ and z is a root of unity times a nonnegative power of t. However since $z(0) = 1$ we must have $z(t)$ identically 1, so that again $S = S_0$. ∎

Let $X^{\#}$ denote the subset of X consisting of points which are not critical points for the fibration π. $X^{\#}$ is then obtained from X by deleting all components of fibers which have multiplicity greater than one, and also deleting all singular points of fibers. For c in C, we denote by $X^{\#}_c$ the fiber of $X^{\#}$ over c.

Let $X^{\#}_0$ denote the subset of $X^{\#}$ obtained by deleting all components of fibers not meeting the zero section S_0. Similarly denote by $X^{\#}_{c0}$ the fiber of $X^{\#}$ over c in C; $X^{\#}_{c0}$ is the component of X_c meeting S_0, minus the singular points of X_c on $X^{\#}_{c0}$.

Note that $X^{\#}_c$ can be made naturally into an abelian group, as follows. Take the germ of the fiber X_c, and let $\mathcal{S}$ be the set of all local sections of π in the germ. Any section must pass through a point of $X^{\#}_c$. The set $\mathcal{S}$ forms a group, by the usual addition of sections: one defines the sum fiber by fiber on the smooth fibers, then closes up the result over the singular fiber. Let $\mathcal{S}_{00}$ be the subgroup of local sections in $\mathcal{S}$ which pass through $S_0 \cap X_c$, i.e., pass through the same point on X_c as does the zero section S_0. The quotient group $\mathcal{S}/\mathcal{S}_{00}$ may be identified with the points of $X^{\#}_c$, and this puts the group structure on $X^{\#}_c$. Note that $X^{\#}_{c0}$ is the connected component of the identity, and $X^{\#}_c/X^{\#}_{c0}$ is a finite abelian group whose order is the number of multiplicity one components of X_c.

The fact that two torsion sections never meet can be expressed as follows:

(VII.3.3)<u>Corollary</u>: For every c in C, the restriction map

$$\text{res}:\text{TMW}(X) \longrightarrow \text{Tors}(X_c^{\#}) \text{ is injective.}$$

We again meed to have some more detailed information about this commutative group $X_c^{\#}$ for the various possibilities of singular fibers X_c. We present the information in the table below.

(VII.3.4)<u>Table</u> of $X_c^{\#}$.

Fiber	$X_{c0}^{\#}$	$X_c^{\#}/X_{c0}^{\#}$
I_0	elliptic	$\{0\}$
$I_{N\geq 1}$	$\mathbb{C}^*$	$\mathbb{Z}/N$
I_N^*	$\mathbb{C}$	$\begin{cases}\mathbb{Z}/2\times\mathbb{Z}/2 & \text{if N is even}\\ \mathbb{Z}/4 & \text{if N is odd}\end{cases}$
II, II^*	$\mathbb{C}$	$\{0\}$
III, III^*	$\mathbb{C}$	$\mathbb{Z}/2$
IV, IV^*	$\mathbb{C}$	$\mathbb{Z}/3$

The column of the $X_{c0}^{\#}$ is quite easy to understand; the last column is interesting in light of Table (VII.2.6). This is no accident:

(VII.3.5)<u>Lemma:</u> Let X_c be a reducible fiber. Then $X_c^{\#}/X_{c0}^{\#} \cong G_{R_c}$.

Proof: This is a local statement, and we may prove it by passing to the germ of the fiber X_c. Let $\mathcal{S}$ be the set of local sections as before, so that $X_c^{\#} \cong \mathcal{S}/\mathcal{S}_{00}$. Let $\mathcal{S}_0$ be the subgroup of $\mathcal{S}$ consisting of sections meeting $X_{c0}^{\#}$. Then $\mathcal{S}_{00} \subset \mathcal{S}_0 \subseteq \mathcal{S}$ and $X_{c0}^{\#} \cong \mathcal{S}_0/\mathcal{S}_{00}$.

The homomorphism γ is defined at the germ level, and gives a map $\gamma:\mathcal{S} \longrightarrow G_{R_c}$; the kernel is precisely $\mathcal{S}_0$. Since any point of $X_c^{\#}$ is hit by some local section, in particular any component of $X_c^{\#}$ is so hit; therefore γ is onto, and so $\mathcal{S}/\mathcal{S}_0 \cong G_{R_c}$. But by the above, this is precisely $X_c^{\#}/X_{c0}^{\#}$. ∎

Lecture VIII: Rational Elliptic Surfaces

1: Extremal rational elliptic surfaces

Let $\pi:X \longrightarrow \mathbb{P}^1$ be a rational elliptic surface with section S_0. In this case $e = e(X) = 12$, and $K_X = -F$. Let $U = \langle S_0, F\rangle$ be the rank two unimodular sublattice of NS(X) generated by S_0 and F; as remarked previously, U splits off NS(X): $NS(X) = U \oplus U^{\perp}$. Since NS(X) is unimodular with signature (1,9) (X is a blow-up of $\mathbb{P}^2$ at nine points), and U is unimodular with signature (1,1), we must have $U^{\perp}$ unimodular with signature (0,8): i.e., $U^{\perp}$ is a negative definite unimodular lattice of rank 8.

The intersection form on $U^{\perp}$, moreover, is even, since $K_X \in U$ and X is rational; hence $U^{\perp}$ is abstractly isomorphic to a lattice of type E_8.

Recall from (VII.2.5) we have $MW(X) \cong U^{\perp}/R$, where R is the sublattice of $U^{\perp}$ generated by components of fibers not meeting S_0.

(VIII.1.1)Definition: An elliptic surface $\pi:X \longrightarrow C$ with section will be called extremal if $\rho = h^{1,1} = 2 + \text{rank}(R)$.

In other words, X is extremal if X has maximal Picard number and the classes of S_0 and components of fibers generate NS(X) over $\mathbb{Q}$. We have an immediate corollary: X is extremal if and only if $\rho = h^{1,1}$ and MW(X) is finite.

In fact, for rational surfaces, the concept of extremal can be viewed in many ways:

(VIII.1.2)Proposition: Let X be a rational elliptic surface with section. Then the following are equivalent:

(a) X is extremal
(b) The relative automorphism group $Aut_C(X)$ is finite.
(c) The number of representations of X as a blow-up of $\mathbb{P}^2$ is finite.
(d) The number of smooth rational curves C with $C^2 < 0$ is finite.
(e) The number of reduced irreducible curves C with $C^2 < 0$ is finite.

Proof: The relative automorphism group $Aut_C(X)$ is the group of automorphisms of X (all of which must preserve the elliptic fibration, since the elliptic fibration is given by $|-K_X|$) which induce the identity automorphism of the base curve C. Let τ be such an automorphism, and consider the image $\tau(S_0)$ of the zero section; it is again a section, and so the automorphism $\tau_{\tau(S_0)}^{-1} \circ \tau$ fixes S_0, where τ_S is the automorphism given by translation by S, for a section S. The automorphisms of X fixing S_0 form a finite group: this is the automorphisms of the generic fiber. Hence $\mathrm{Aut}_C(X)$ is finite if and only if the group of sections MW(X) is finite; this proves that (a) and (b) are equivalent. The implications (e) ⇒ (d) ⇒ (c) are obvious. Since $K_X = -F$, a smooth rational curve E on X is exceptional if and only if it is a section, proving (a) ⇔ (d), since in any case a smooth rational curve C must satisfy $-2 = C^2 + CK$, or $C^2 = CF - 2 \geq -2$, and the (-2)-curves are always finite in number: they are the components of reducible fibers. If C is reduced and irreducible with $C^2 < 0$, then we must have $-2 \leq C^2 + CK \leq -1$, forcing C to be smooth rational and either a (-1)- or (-2)-curve; this shows that (d) and (e) are equivalent. Finally, since any rational elliptic surface is a blow-up of $\mathbb{P}^2$, we have (c) ⇒ (d). ■

As consequences of extremality for rational elliptic surfaces, we have the following.

(VIII.1.3)<u>Proposition:</u> Let $\pi: X \longrightarrow \mathbb{P}^1$ be an extremal rational elliptic surface with section. Then:

(a) $\mathrm{disc}(R) = \prod_F d(F) = |MW(X)|^2$, and in particular is a perfect square.

(b) $\sum_F (e - r) = 4$

(c) X has, for singular fibers, either:

- o 4 semistable fibers
- o 3 singular fibers, exactly 2 of them semistable

or o 2 unstable singular fibers.

Proof: The lattice R is the orthogonal direct sum of the lattices R_F for each fiber F, and the discriminant of R_F is d(F) by definition. Hence $\mathrm{disc}(R) = \prod d(F)$ is obvious. By general lattice theory, $\mathrm{disc}(R) = \mathrm{disc}(U^{\perp}) \cdot [U^{\perp}:R]^2$, and since $MW \cong U^{\perp}/R$, and $U^{\perp}$ is unimodular, we have that $\mathrm{disc}(R) = |MW(X)|^2$. Since $e = \sum_F e_F = 12$ and $\mathrm{rank}(R) = \sum r_F = 8$, we have (b). Finally, (c) follows from Lemma (IV.3.2)(b). ■

This Proposition allows us to classify all configurations of singular fibers on extremal rational elliptic surfaces. We give the list in the following table.

(VIII.1.4) Table of possible configurations of singular fibers on extremal rational elliptic surfaces.

Singular fibers	degree(J)	$\lvert MW(X) \rvert$	Notation
II, II^*	0	1	X_{22}
III, III^*	0	2	X_{33}
IV, IV^*	0	3	X_{44}
I_0^*, I_0^*	0	4	$X_{11}(j)$, $j \in \mathbb{C}$
II^*, I_1, I_1	2	1	X_{211}
III^*, I_2, I_1	3	2	X_{321}
IV^*, I_3, I_1	4	3	X_{431}
I_4^*, I_1, I_1	6	2	X_{411}
I_1^*, I_4, I_1	6	2	X_{141}
I_2^*, I_2, I_2	6	4	X_{222}
I_9, I_1, I_1, I_1	12	3	X_{9111}
I_8, I_2, I_1, I_1	12	4	X_{8211}
I_6, I_3, I_2, I_1	12	6	X_{6321}
I_5, I_5, I_1, I_1	12	5	X_{5511}
I_4, I_4, I_2, I_2	12	8	X_{4422}
I_3, I_3, I_3, I_3	12	9	X_{3333}

Proof: There are three cases, corresponding to the number of singular fibers.

Case of two unstable singular fibers.

Since $\sum e_F = 12$ we have the four possibilities above, plus the configurations $\{I_2^*, IV\}$, $\{I_3^*, III\}$, and $\{I_4^*, II\}$. The first two violate Proposition (VIII.1.3)(a) and the third leads to degree(J) < 0 by Proposition (IV.4.8). An alternate argument can be given by remarking that since there is no non-constant holomorphic map from $\mathbb{C}^*$ to the upper half-plane, J must be constant; however at fibers of type I_N^* J has a pole.

Case of three singular fibers, exactly two semistable and one unstable.

Again using $\sum e_F = 12$, Proposition (VIII.1.3)(a), and Proposition (IV.4.8), we are left only with the configurations in the table.

Case of four semistable singular fibers.

The possibilities are to have $\{I_{n_1}, I_{n_2}, I_{n_3}, I_{n_4}\}$ with $\sum n_i = 12$ and $\prod n_i$ equal to a perfect square; this gives only the six sets in the table. ■

The following theorem is proved in [MP].

(VIII.1.5)Theorem: For every configuration of possible singular fibers in the Table (VIII.1.4), there is a unique extremal rational elliptic surface with section with that configuration of singular fibers, except for the configuration $\{I_0^*, I_0^*\}$; these are classified by their j-invariant, which must be some constant $j \in \mathbb{C}$, and can be any complex number.

The 6 semistable surfaces above were studied by Beauville [B], and some authors have called these surfaces Beauville surfaces.

2: A deformation result

Let $\pi : X \longrightarrow \mathbb{P}^1$ be a semistable elliptic surface with section, i.e., all fibers of π are of type I_n. Assume that X has s singular fibers, which are of types $I_{n_1}, \ldots, I_{n_s}$. In this case we will say that X realizes the unordered s-tuples $[n_1, \ldots, n_s]$; note that $\sum n_i$ is a multiple of 12: it is e(X). Conversely, given a set $[n_1, \ldots, n_s]$ of s positive integers whose sum is divisible by 12 (repetitions are allowed), we will say that $[n_1, \ldots, n_s]$ exists as a semistable elliptic surface over $\mathbb{P}^1$, or that simply the set exists, if there is a semistable elliptic surface with section over $\mathbb{P}^1$ with exactly s singular fibers of types $I_{n_1}, \ldots, I_{n_s}$.

For example, the Beauville rational elliptic surfaces realize [9,1,1,1], [8,2,1,1], [6,3,2,1], [5,5,1,1], [4,4,2,2], and [3,3,3,3]; these six 4-tuples exist.

It is our goal in this section to prove the following.

(VIII.2.1)Lemma: Assume that $[n_1, \ldots, n_s]$ exists.
Then $[n_1, \ldots, n_{i-1}, a, b, n_{i+1}, \ldots, n_s]$ exists, for any $a, b \geq 1$ with $a+b = n_i$.

Proof: By pulling back the surface with J = identity, one sees that to prove that $[m_1, \ldots, m_s]$ exists it suffices to construct an appropriate J-map, properly ramified over 0, 1, and ∞. To obtain only semistable fibers of types $I_{m_1}, \ldots, I_{m_s}$, one must have $\sum m_i = 12k$ for some $k \geq 1$, and the J-map must have exactly s points over ∞, with multiplicities $m_1, \ldots, m_s$; in addition, one must have the proper ramification over 0 and 1.

If $[n_1, \ldots, n_s]$ exists, then a J-map exists, properly ramified over 0 and

1, with s points over ∞, with multiplicities $n_1,\dots,n_s$. Such a J-map is determined by its monodromy representation, given by a set of permutations $\{\sigma_0,\sigma_1,\sigma_\infty,\tau_1,\dots,\tau_r\}$, such that their product is the identity in S_{12k} and they generate a transitive subgroup: σ_α is the monodromy around α, and the τ's are the monodromy around the other branch points unequal to 0, 1, or ∞, if any. The cycle structure of σ_∞ must be given by s cycles, of lengths $n_1,\dots,n_s$. Because of the identity

$$(1,2,3,\dots,n) = [(1,2,\dots,a)(a+1,\dots,n)]\cdot[(a,n)]$$

we may replace the n_i-cycle in σ_∞ by the product of the two cycles of lengths a and b, at the expense of adding an extra permutation (and therefore an extra simple ramification point over a new branch point to J). This new list of permutations still has product the identity, and still generates a transitive subgroup of S_{12k}; hence it defines a J-map, properly ramified over 0, 1, and ∞, to produce an elliptic surface which realizes the (s+1)-tuple $[n_1,\dots,n_{i-1},a,b,n_{i+1},\dots,n_s]$. ■

3: Semistable rational elliptic surfaces

As a corollary of Lemma (VIII.2.1) and the existence of the six 4-tuples obtained by the Beauville surfaces, we have the following.

(VIII.3.1)<u>Corollary</u> The following s-tuples exist.

s = 4: $91^3, 821^2, 6321, 5^21^2, 4^22^2, 3^4$

s = 5: $81^4, 721^3, 631^3, 541^3, 62^21^2, 5321^2, 4^221^2, 432^21, 42^4, 3^321$

s = 6: $71^5, 621^4, 531^4, 52^21^3, 4^21^4, 4321^3, 42^31^2, 3^31^3, 3^22^21^2, 32^41, 2^6$

s = 7: $61^6, 521^5, 431^5, 42^21^4, 3^221^4, 32^31^3, 2^51^2$

s = 8: $51^7, 421^6, 3^21^6, 32^21^5, 2^41^4$

s = 9: $41^8, 321^7, 2^31^6$

s = 10: $31^9, 2^21^8$

s = 11: 21^{10}

s = 12: 1^{12}

Note that every s-tuple with $s \geq 6$ and $\sum n_i = 12$ is on the above list, i.e., can be obtained from the six 4-tuples:

(VIII.3.2)<u>Corollary</u> Let $[n_1,\dots,n_s]$ be an s-tuple with $\sum n_i = 12$. Then if $s \geq 6$, $[n_1,\dots,n_s]$ exists.

Our goal is to prove that the list of Corollary (VIII.3.1) is complete, i.e., that these are the only s-tuples with $\sum n_i = 12$ which exist. In view of the previous corollary, the necessity that s be at least 4, and the classification in the case s = 4, it suffices to show the following.

(VIII.3.3)<u>Proposition:</u> The three 5-tuples

[5,2,2,2,1], [4,3,3,1,1], and [3,3,2,2,2]

do not exist.

We will take this up in the next lecture.

Lecture IX: Some Lattice Theory

1: Generalities on discriminant form groups

Let L be a finitely generated free $\mathbb{Z}$-module, and let $<-,->$ be an even non-degenerate $\mathbb{Z}$-valued symmetric bilinear form on L. $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ naturally inherits the bilinear form, which will still be non-degenerate and symmetric; moreover $L \subset L_{\mathbb{Q}}$ naturally. Define $L^{\#} = \{x \in L_{\mathbb{Q}} | <x,\ell> \in \mathbb{Z} \ \ \forall \ell \in L\}$. We of course have $L \subseteq L^{\#}$, since the form is $\mathbb{Z}$-valued on L. The natural map $\phi: L^{\#} \longrightarrow L^{*}$ $(= \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z}))$ defined by $\phi(x) = <x,->$ is an isomorphism, and so we see that $L^{\#}$ is a free $\mathbb{Z}$-module with the same rank as L; in particular, L has finite index in $L^{\#}$.

Define $G_L = L^{\#}/L$, the so-called <u>discriminant-form group</u> of L. Its order is the absolute value of the discriminant of L,

(IX.1.1) $$|G_L| = |\mathrm{disc}(L)|,$$

since both sides are computed as the absolute value of the determinant of any matrix for $<-,->$ on a $\mathbb{Z}$-basis of L. If we define the <u>length</u> $\ell(G)$ of a finite abelian group to be the minimum number of generators of G, we also have that

(IX.1.2) $$\ell(G_L) \le \mathrm{rank}(L),$$

since G_L is generated by the cosets of the rank(L) generators of $L^{\#}$.

One can define a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form q_L, the <u>discriminant-form</u>, on G_L by setting, for x in $L^{\#}$, $q_L(x) = \frac{1}{2}<x,x> \bmod \mathbb{Z}$. The reader should check that q_L is well-defined, and satisfies $q_L(nx) = n^2 q_L(x)$ for $n \in \mathbb{Z}$ and $x \in G_L$. Moreover, the function $q_L(x+y)-q_L(x)-q_L(y)$ is exactly the induced symmetric bilinear form $<-,->$ on G_L, with values in $\mathbb{Q}/\mathbb{Z}$.

(IX.1.3)<u>Example:</u> Let L be the lattice of rank N-1 representing A_{N-1}. Then L is realized as the lattice R_c, where X_c is a fiber of type I_N on al elliptic surface with section. In particular, as we noted in Table (VII.2.6), $G_L \cong \mathbb{Z}/N\mathbb{Z}$. A generator for G_L is afforded by the coset of the element $e_1^{\#} = \frac{-1}{N}\left[\sum_{i=1}^{N-1} ie_i\right]$, where $\{e_i\}$ is the natural basis of L, namely the classes of the components of the cycle I_N; these are numbered so that e_i meets $e_{i\pm1}$ around the cycle, and e_0 meets the zero section. The element $e_1^{\#}$ meets e_1

exactly once, and meets no other e_i; its image in L^* is the dual element to e_1.

A calculation gives that $q_L(e_1^\# \bmod L) = (1-N)/2N$, so that if we identify G_L with $\mathbb{Z}/N$ by sending $e_1^\# \bmod L$ to 1, we have that

$$\text{(IX.1.4)} \qquad q_{A_{N-1}}(a) = a^2(1-N)/2N.$$

It will be useful later to remark the following:

(IX.1.5) For the following lattices L, G_L has no nonzero q_L-isotropic elements (i.e., elements g with $q_L(g) = 0$): A_4, A_3, $A_2 \oplus A_2$, $A_1 \oplus A_1 \oplus A_1$.

One of the main applications of this discriminant-form construction is to the analysis of embeddings of lattices. The following is a typical example.

(IX.1.6)<u>Lemma:</u> There is a 1-1 correspondence between

$$\left\{ \begin{array}{l} \text{intermediate lattices } M \\ L \subseteq M \subseteq L^\#, \text{ such that} \\ \langle -,- \rangle|_M \text{ is } \mathbb{Z}\text{-valued and even} \end{array} \right\} \text{ and } \left\{ \begin{array}{l} q_L\text{-isotropic subgroups} \\ H \subseteq G_L \end{array} \right\}$$

Moreover if M corresponds to H, then $G_M \cong H^\perp/H$, and q_M is induced from q_L.

Proof: Of course by q_L-isotropic I mean that $q_L(h) = 0$ for every h in H. The lemma is easily proved; the correspondence is the usual one, sending an intermediate lattice M to M/L, and a subgroup H to $\pi^{-1}(H)$, where $\pi: L^\# \longrightarrow G_L$ is the natural quotient map. We must check that the sets above correspond. Assume that M is an intermediate lattice such that $\langle -,- \rangle|_M$ is even and $\mathbb{Z}$-valued; then clearly $q_L(m) = \langle m,m \rangle/2 \bmod \mathbb{Z} = 0$, so that M/L is q_L-isotropic. Conversely, assume that H is q_L-isotropic, and let m and n be in $\pi^{-1}(H)$. Then $\langle m,n \rangle \bmod \mathbb{Z} = q_L(m+n) - q_L(m) - q_L(n) = 0$, so $\langle m,n \rangle \in \mathbb{Z}$, and the form is $\mathbb{Z}$-valued on M; moreover, since $\langle m,m \rangle = q_L(2m) - 2q_L(m) = 2q_L(m)$, we have evenness also.

To prove the last statement, one must simply show that $M^\# = \pi^{-1}(H^\perp)$, which is obvious. ∎

The following will be useful.

(IX.1.7)<u>Lemma:</u> Suppose that U is a unimodular lattice, and L_1 and L_2 are two nondegenerate sublattices of U, such that $L_1 = L_2^{\perp}$ and $L_2 = L_1^{\perp}$. Then there exists an isomorphism between G_{L_1} and G_{L_2} such that $q_{L_1} = -q_{L_2}$ under the isomorphism.

Proof: The nondegeneracy of the L_i implies that $L_1 \oplus L_2$ is a sublattice of U; hence we may view U as an intermediate lattice between $L_1 \oplus L_2$ and $(L_1 \oplus L_2)^{\#}$, so that there exists a q-isotropic subgroup H of $G_{L_1 \oplus L_2}$ corresponding to U. Since U is unimodular, G_U is trivial, so that $H^{\perp} = H$ by the previous lemma. Note that since the L_i are orthogonal, $G_{L_1 \oplus L_2} \cong G_{L_1} \oplus G_{L_2}$. Let π_i be the projection of $G_{L_1 \oplus L_2}$ onto G_{L_i}.

<u>claim:</u> $\pi_i|_H : H \longrightarrow G_{L_i}$ is an isomorphism for both i.

<u>Why</u>: First let us show injectivity: suppose that $\pi_1(h) = 0$, for $h \in H$. Then $h = (0, g_2)$ for some g_2 in G_{L_2}. Hence there exists an element u in U, mapping to h, of the form $(0, x_2)$, where $x_2 \in L_2^{\#}$ and $g_2 = x_2 \bmod L_2$. Since $u \in L_1^{\perp}$, we must have $u \in L_2$, so that $x_2 \in L_2$ and $g_2 = 0$, whence $h = 0$. Therefore π_1 is injective on H; the argument for π_2 is the same.

The injectivity shows that $|H| \le |G_{L_i}|$ for both i; since $H^{\perp} = H$, and the order of $H^{\perp}$ is the index of H (this is a general fact), we have that $|H|^2 = |G_{L_1 \oplus L_2}| = |G_{L_1}| \cdot |G_{L_2}|$. Hence $|H| = |G_{L_1}| = |G_{L_2}|$ and the injectivity also implies surjectivity. This proves the claim.

To finish the proof, define $f : G_{L_1} \longrightarrow G_{L_2}$ by $f = (\pi_2|_H) \circ (\pi_1|_H)^{-1}$; f is an isomorphism by the claim. If $h \in H$, then, since H is isotropic, we have $0 = q_{L_1 \oplus L_2}(h) = q_{L_1}(\pi_1(h)) + q_{L_2}(\pi_2(h))$, proving that the quadratic forms for the L_i are opposite in sign. ∎

(IX.1.8)<u>Corollary</u> Suppose L is a nondegenerate sublattice of a unimodular lattice U. Then $G_{L^{\perp\perp}} \cong G_{L^{\perp}}$ and $q_{L^{\perp\perp}} = -q_{L^{\perp}}$.

Proof: Just apply the previous lemma to $L^{\perp}$ and $L^{\perp\perp}$. ∎

2: The 3 impossible 5-tuples

We are now in a position to prove Proposition (VIII.3.3), i.e., to prove that the three 5-tuples [5,2,2,2,1], [4,3,3,1,1], and [3,3,2,2,2] do not exist.

Suppose that $\pi:X \longrightarrow C$ is a semistable rational elliptic surface with section, realizing the s-tuple $[n_1,\ldots,n_s]$; i.e., there are exactly s singular fibers of types $I_{n_1},\ldots,I_{n_s}$, and $\sum n_i = 12$. Let R be the sublattice of NS(X) generated by the components of fibers not meeting the zero section S_0; R is a lattice of rank $\sum(n_i-1) = 12 - s$, isomorphic to $\oplus_i A_{n_i-1}$. Let U be the lattice generated by S_0 and the fiber F; NS(X) is unimodular (X is rational), so is $U^{\perp}$, and since $K_X \in U$, $U^{\perp}$ is even; in fact $U^{\perp}$ is isomorphic to the E_8 lattice, but we do not need to know that. In any case $U^{\perp}$ has rank 8, since U has rank 2 and NS(X) has rank 10. Therefore:

(IX.2.1) If $[n_1,\ldots,n_s]$ exists, then $\oplus_i A_{n_i-1}$ embeds into a unimodular lattice of rank 8.

Now suppose further that s = 5; then rank(R) = 7, so that $K = R^{\perp}$ in $U^{\perp}$ has rank 1. Therefore G_K is cyclic, and by Corollary (IX.1.8), so is $G_{R^{\perp\perp}}$. The inclusion $R \subseteq R^{\perp\perp}$ is between lattices of the same rank, so that we can view $R^{\perp\perp}$ as an intermediate lattice between R and $R^{\#}$; therefore $R^{\perp\perp}$ corresponds to an isotropic subgroup H of G_R with $G_{R^{\perp\perp}} \cong H^{\perp}/H$. Thus:

(IX.2.2) If $[n_1,\ldots,n_5]$ exists, there is an isotropic subgroup H of G_R with $H^{\perp}/H$ cyclic.

In particular:

(IX.2.3) If $G_{\oplus_i A_{n_i-1}}$ is not cyclic, and has no nonzero isotropic elements, then $[n_1,\ldots,n_5]$ does not exist.

The final ingredient is provided by the next lemma.

(IX.2.4)Lemma: For the lattices

$A_1 \oplus A_1 \oplus A_1 \oplus A_2 \oplus A_2$, $A_1 \oplus A_1 \oplus A_1 \oplus A_4$, and $A_2 \oplus A_2 \oplus A_3$,
the discriminant-form groups have no nonzero isotropic elements.

Proof: Since (2,3), (2,5), and (3,4) are relatively prime, any isotropic element of these lattices must decompose into isotropic elements of the summands $A_1 \oplus A_1 \oplus A_1$, $A_2 \oplus A_2$, A_3, and A_4: no cancellation is possible. This forces any isotropic element to be zero by (IX.1.5). ∎

Since the discriminant-form groups for the three lattices above are not cyclic, applying (IX.2.4) to (IX.2.3) proves Proposition (VIII.3.3).

3: The Length and Discriminant Criteria for deducing torsion in MW

In the rest of this lecture we want to develop some simple criteria for deducing the existence of torsion in the Mordell-Weil group of a semistable elliptic surface over $\mathbb{P}^1$. The first is the "Length Criterion":

(IX.3.1)Proposition: Assume $\pi: X \longrightarrow \mathbb{P}^1$ realizes $[n_1, \ldots, n_s]$. Fix a prime number p. If p divides s-3 or more of the n_i's, then there is p-torsion in MW(X).

Proof: Let R be the lattice of components of fibers not meeting the zero section S_0, and let $L = \langle S_0, F \rangle \oplus R$. The assumption implies that the p-length of G_R is at least s-3. Considering L as a sublattice of $H^2(X,\mathbb{Z})$ (mod torsion), we have that

$\text{rank}(L^\perp) = h^2(X,\mathbb{Z}) - \text{rank}(L) = h^2(X,\mathbb{Z}) - (2 + \text{rank}(R))$

$= (12\chi - 2) - 2 - \sum(n_i - 1) = s - 4;$

hence the length of $G_{L^\perp}$ = length($G_{R^\perp}$) $\leq$ s-4.

Now $R \subseteq R^{\perp\perp}$ and this inclusion corresponds to a totally isotropic subgroup H of G_R; moreover, $R^{\perp\perp}/R = H$ is isomorphic to TMW(X), the torsion in the Mordell-Weil group. Assume then that there is no p-torsion in MW(X); then there is no p-torsion in $R^{\perp\perp}/R = H$; hence the p-length of $H^\perp/H$ equals the p-length of G_R. Since $H^\perp/H \cong G_{R^{\perp\perp}}$, we have that

the p-length of $G_{R^{\perp\perp}}$ = p-length of $G_R \geq$ s-3.

Now we apply (IX.1.8) to obtain a contradiction: since $G_{R^\perp}$ and $G_{R^{\perp\perp}}$ are isomorphic, certainly they have the same p-length; thus $s - 3 \le \text{p-length}(G_{R^{\perp\perp}}) \le \text{length}(G_{R^{\perp\perp}}) = \text{length}(G_{R^\perp}) \le s - 4$. This contradiction proves the result. ■

The length Criterion for deducing p-torsion in the Mordell-Weil group can be sharpened to apply to the extreme case, when a prime p divides exactly s-4 of the n_i's. For this application, one must use the discriminant of G_R also, not just the length; for this reason we call it the "Discriminant Criterion":

(IX.3.2)<u>Proposition:</u> Assume $\pi:X \longrightarrow \mathbb{P}^1$ realizes $[n_1,\ldots,n_s]$. Fix a prime number p, and assume that p does not divide $n_1,\ldots,n_4$, but p does divide n_i for $i \ge 5$.

(a) Assume $p = 2$ and 4 divides n_i for $i \ge 5$. If

$(-1)^s n_1n_2n_3n_4 \not\equiv \prod_{i\ge5}(n_i-1) \bmod 8$, then there is 2-torsion in MW(X).

(b) Assume p is odd, and $n_1n_2n_3n_4$ is not a square modulo p. Then there is p-torsion in MW(X).

Proof: Write $n_i = m_ip^{e_i}$ for $i \ge 5$, with $p\nmid m_i$. Assume that there is no p-torsion in MW(X). Let H be the isotropic subgroup of G_R corresponding to $R^{\perp\perp}$; H is isomorphic to TMW(X), and the assumption of no p-torsion in MW(X) implies that $p\nmid|H|$. Again write $L = \langle S_0,F\rangle \oplus R$; then $|G_{L^\perp}| = |G_{L^{\perp\perp}}| = |G_{R^{\perp\perp}}| = |G_R|/|H|^2 = \prod_{i\ge1} n_i/|H|^2$. Since $L^\perp$ has signature (2k,s-4-2k) for some k, there are s-4-2k negative eigenvalues for the matrix for $L^\perp$; since $|G_{L^\perp}|$ is the absolute value of the discriminant of $L^\perp$, we have that $\operatorname{disc}(L^\perp) = (-1)^s \prod_{i\ge1} n_i/|H|^2$.

Let us write $G^{(p)}$ for the p-part of a finite abelian group G. Then

$$\begin{aligned} G^{(p)}_{L^\perp} &\cong G^{(p)}_{L^{\perp\perp}} \quad \text{(by (IX.1.8))} \\ &\cong G_R^{(p)} \quad \text{(since } |H| \text{ is prime to p)} \\ &\cong \prod_{i=5}^{s} \mathbb{Z}/p^{e_i}\mathbb{Z}. \end{aligned}$$

Moreover, $q^{(p)}_{L^\perp}(x_5 \bmod p^{e_5},\dots,x_s \bmod p^{e_s}) = -q^{(p)}_{L^{\perp\perp}}(x_5,\dots,x_s)$
$= -q^{(p)}_R(0,0,0,0,m_5x_5 \bmod n_5,\dots,m_sx_s \bmod n_s)$
$= -\sum_{j=5}^{s}[(1-n_j)/(2n_j)](m_jx_j)^2$ (using (IX.1.4))
$= \sum_{j=5}^{s}[(n_j-1)m_j/2p^{e_j}]x_j^2$;
in particular, the discriminant form of $G^{(p)}_{L^\perp}$ diagonalizes. Therefore, since $\mathrm{rank}(L^\perp)$ = p-length of $G_{L^\perp}$, the p-adic form on $L^\perp\otimes\mathbb{Z}_p$ also diagonalizes (here we use the assumption that 4 divides the n_i's in case $p = 2$). Hence there is a basis for $L^\perp\otimes\mathbb{Z}_p$ over $\mathbb{Z}_p$ such that the matrix of the bilinear form is $\mathrm{diag}(p^{e_5}/(n_5-1)m_5,\dots,p^{e_s}/(n_s-1)m_s)$; these are the eigenvalues that produce the above formula for $q^{(p)}_{L^\perp}$, and part of the p-adic theory is that the form on $L^\perp\otimes\mathbb{Z}_p$ is unique under our hypotheses. The eigenvalues are well-defined modulo squares of units in $\mathbb{Z}_p$, so that we may equally well take $\mathrm{diag}(p^{e_5}(n_5-1)m_5,\dots,p^{e_s}(n_s-1)m_s) = \mathrm{diag}(n_5(n_5-1),\dots,n_s(n_s-1))$ as the matrix. In this case we calculate $\mathrm{disc}(L^\perp\otimes\mathbb{Z}_p)\ \prod_{j=5}^{s} n_j(n_j-1) \bmod (\mathbb{Z}_p^\times)^2$. Since the discriminant of $L^\perp\otimes\mathbb{Z}_p$ is induced by the discriminant of $L^\perp$, we have from our previous calculation that $(-1)^s\prod_{i\geq1} n_i/|H|^2 = \prod_{j=5}^{s} n_j(n_j-1) \bmod (\mathbb{Z}_p^\times)^2$, or equivalently that $(-1)^s n_1n_2n_3n_4 = \prod_{j=5}^{s}(n_j-1) \bmod (\mathbb{Z}_p^\times)^2$. If $p = 2$, equality mod $(\mathbb{Z}_2^\times)^2$ is measured by equality modulo 8, proving (a). If p is odd, then equality mod $(\mathbb{Z}_p^\times)^2$ is measured by equality modulo squares mod p. Since $\prod_{j=5}^{s}(n_j-1) = (-1)^s \bmod p$, after canceling we obtain the denial of the condition of (b). ■

Lecture X: Configurations of fibers on semistable K3 surfaces

X.1: The statement of the results

In this last lecture I want to outline the results of [MP2] on the possible configurations of singular fibers on semistable elliptic K3 surfaces. The goal is to prove a result analogous to Corollary (VIII.3.1). There all realizable s-tuples $[n_i]$ with $\sum n_i = 12$ were found: this was the rational elliptic case. Now we focus on the next case, namely the s-tuples with $\sum n_i = 24$, which is the case of semistable elliptic K3 surfaces.

There are many more combinatorial possibilities now, and for an exhaustive analysis the reader should consult [MP2]. In this lecture I will outline the techniques used and state the results.

Suppose that $\pi : X \longrightarrow \mathbb{P}^1$ realizes $[n_1,\ldots,n_s]$ with $\sum n_i = 24$. An application of Proposition (IV.4.8) shows that $x = -2 + \frac{1}{6}\left[6\sum_{m\geq 1}(i_m) - 24\right] = s - 6$, where x is the "extra" ramification of the J-map; recall that $x \geq 0$, and moreover $x = 0$ if and only if every fiber with $J = 0$ has $m = 3$, every fiber with $J = 1$ has $m = 2$, and the only ramification of J occurs over 0, 1, and ∞. We see then that s is at least 6, and $s = 6$ if and only if the above conditions on the multiplicities of J hold.

There are 199 6-tuples $[n_1,\ldots,n_6]$ with $\sum n_i = 24$. The first result is that 112 of them exist:

(X.1.1) <u>Theorem:</u> The following 112 6-tuples exist.

[1,1,1,1,1,19]	[1,1,1,1,2,18]	[1,1,1,1,3,17]	[1,1,1,1,4,16]
[1,1,1,1,5,15]	[1,1,1,1,6,14]	[1,1,1,1,7,13]	[1,1,1,1,9,11]
[1,1,1,1,10,10]	[1,1,1,2,2,17]	[1,1,1,2,3,16]	[1,1,1,2,4,15]
[1,1,1,2,5,14]	[1,1,1,2,6,13]	[1,1,1,2,7,12]	[1,1,1,2,8,11]
[1,1,1,2,9,10]	[1,1,1,3,3,15]	[1,1,1,3,4,14]	[1,1,1,3,5,13]
[1,1,1,3,6,12]	[1,1,1,3,7,11]	[1,1,1,3,8,10]	[1,1,1,4,6,11]
[1,1,1,4,7,10]	[1,1,1,5,5,11]	[1,1,1,5,6,10]	[1,1,1,5,7,9]
[1,1,1,6,7,8]	[1,1,1,7,7,7]	[1,1,2,2,2,16]	[1,1,2,2,3,15]
[1,1,2,2,4,14]	[1,1,2,2,5,13]	[1,1,2,2,6,12]	[1,1,2,2,7,11]
[1,1,2,2,9,9]	[1,1,2,3,3,14]	[1,1,2,3,4,13]	[1,1,2,3,5,12]
[1,1,2,3,6,11]	[1,1,2,3,7,10]	[1,1,2,3,8,9]	[1,1,2,4,4,12]
[1,1,2,4,5,11]	[1,1,2,4,6,10]	[1,1,2,4,7,9]	[1,1,2,4,8,8]
[1,1,2,5,5,10]	[1,1,2,5,6,9]	[1,1,2,5,7,8]	[1,1,2,6,6,8]
[1,1,3,3,4,12]	[1,1,3,3,5,11]	[1,1,3,3,8,8]	[1,1,3,4,4,11]
[1,1,3,4,6,9]	[1,1,3,4,7,8]	[1,1,3,5,6,8]	[1,1,3,5,7,7]
[1,1,4,4,7,7]	[1,1,4,5,6,7]	[1,1,4,6,6,6]	[1,1,5,5,6,6]
[1,2,2,2,3,14]	[1,2,2,2,5,12]	[1,2,2,2,7,10]	[1,2,2,3,3,13]
[1,2,2,3,4,12]	[1,2,2,3,5,11]	[1,2,2,3,6,10]	[1,2,2,3,7,9]
[1,2,2,4,5,10]	[1,2,2,4,7,8]	[1,2,2,5,5,9]	[1,2,2,5,6,8]
[1,2,2,6,6,7]	[1,2,3,3,3,12]	[1,2,3,3,4,11]	[1,2,3,3,6,9]
[1,2,3,3,7,8]	[1,2,3,4,4,10]	[1,2,3,4,5,9]	[1,2,3,4,6,8]
[1,2,3,5,6,7]	[1,2,4,4,6,7]	[1,2,4,5,5,7]	[1,2,4,5,6,6]
[1,3,3,3,5,9]	[1,3,3,4,5,8]	[1,3,3,5,6,6]	[1,3,4,4,4,8]
[1,3,4,4,5,7]	[2,2,2,2,8,8]	[2,2,2,3,3,12]	[2,2,2,3,5,10]
[2,2,2,4,6,8]	[2,2,2,6,6,6]	[2,2,3,3,4,10]	[2,2,3,3,7,7]
[2,2,3,4,5,8]	[2,2,3,5,5,7]	[2,2,4,4,4,8]	[2,2,4,4,6,6]
[2,2,5,5,5,5]	[2,3,3,3,4,9]	[2,3,3,4,5,7]	[2,3,3,4,6,6]
[2,3,4,4,5,6]	[3,3,3,3,6,6]	[3,3,4,4,5,5]	[4,4,4,4,4,4]

I will discuss the proof of this theorem in the next section. One should view this list as the starting point for the list of all realizable s-tuples $[n_i]$ with $\sum n_i = 24$, using the deformation result, namely Lemma (VIII.2.1). A relatively simple calculation with the 112 6-tuples above gives the following.

(X.1.2)<u>Corollary:</u> All 7-tuples except the 34 below exist.

[1,1,1,4,4,4,9]	[1,1,1,4,4,5,8]	[1,1,3,3,3,3,10]	[1,1,3,3,3,6,7]
[1,1,3,4,5,5,5]	[1,1,4,4,4,4,6]	[1,1,4,4,4,5,5]	[1,2,2,2,2,2,13]
[1,2,2,2,2,4,11]	[1,2,2,2,2,6,9]	[1,2,2,2,4,4,9]	[1,2,3,3,5,5,5]
[1,2,4,4,4,4,5]	[1,3,3,3,3,3,8]	[1,3,3,3,3,4,7]	[1,3,3,3,4,4,6]
[2,2,2,2,2,2,12]	[2,2,2,2,2,3,11]	[2,2,2,2,2,4,10]	[2,2,2,2,2,5,9]
[2,2,2,2,2,7,7]	[2,2,2,2,3,4,9]	[2,2,2,2,3,6,7]	[2,2,2,2,4,5,7]
[2,2,2,2,5,5,6]	[2,2,2,3,4,4,7]	[2,2,2,4,4,5,5]	[2,2,3,3,3,3,8]
[2,2,3,3,3,5,6]	[2,3,3,3,3,3,7]	[2,3,3,3,3,5,5]	[2,3,3,4,4,4,4]
[3,3,3,3,3,4,5]	[3,3,3,3,4,4,4]		

(X.1.3)<u>Corollary:</u> All 8-tuples except the 11 below exist.

[1,1,1,4,4,4,4,5]	[1,1,3,3,3,3,3,7]	[1,2,2,2,2,2,2,11]
[1,2,2,2,2,2,4,9]	[1,3,3,3,3,3,4,4]	[2,2,2,2,2,2,2,10]
[2,2,2,2,2,2,3,9]	[2,2,2,2,2,2,5,7]	[2,2,2,2,2,3,4,7]
[2,2,2,2,2,4,5,5]	[2,2,3,3,3,3,3,5]	

(X.1.4)<u>Corollary:</u> All 9-tuples except the 3 below exist.

[1,2,2,2,2,2,2,2,9] [2,2,2,2,2,2,2,3,7] [2,2,2,2,2,2,2,5,5]

(X.1.5)<u>Corollary:</u> All s-tuples with $s \geq 10$ exist.

As mentioned above, the proof of Corollaries (X.1.2)-(X.1.5) are straightforward computations with the deformation Lemma. One simple remark should be made: once one has all 10-tuples, then clearly all s-tuples with s at least 10 can be obtained, so one only needs to verify the statements up to the $s = 10$ level.

It will turn out that we have generated all of the realizable s-tuples $[n_i]$ with $\sum n_i = 24$ with Theorem (X.1.1) and its Corollaries. I.e., none of the remaining 87 6-tuples exist, and none of the 7-, 8-, or 9- tuples listed in the Corollaries exist.

X.2: The proofs of existence

In this section we will prove Theorem (X.1.1), and "construct" elliptic K3 surfaces realizing the 112 6-tuples listed there. To construct one of these, it suffices to construct its J-map: the elliptic surface is then obtained by pulling back one of the surfaces with J = identity, and performing some quadratic twists to remove *-fibers.

To construct the J-map, note that in the case of the 112 6-tuples, J has degree 24 and is branched only over 0, 1, and ∞. Moreover, every inverse image of $J = 0$ has multiplicity 3, every inverse image of $J = 1$ has multiplicity 2, and the six inverse images of $J = \infty$ have multiplicities equal to the 6 integers $\{n_i\}$. Upon choosing a base point in $\mathbb{P}^1$, the monodromy around 0 for the covering is a permutation σ_0 in S_{24} with cycle structure 3^8; the monodromy around 1 is a permutation σ_1 in S_{24} with cycle structure 2^{12}; and the monodromy around ∞ is a permutation σ_∞ in S_{24} with cycle structure $(n_1,\dots,n_6)$. These permutations satisfy $\sigma_0 \cdot \sigma_1 \cdot \sigma_\infty$ = identity, and they generate a transitive subgroup of S_{24}.

Conversely, if one can find three such permutations in S_{24}, one builds the covering with covering map J in the usual way, and Hurwitz's formula along with the transitivity ensures that the cover has genus 0; the cycle structures guarantee that the ramification is as desired. Therefore:

(X.2.1)<u>Lemma:</u> A 6-tuple $[n_i]$ with $\sum n_i = 24$ exists if and only if there exist three permutations σ_0, σ_1, and σ_∞ in S_{24} such that

(a) the cycle structure of σ_0 is 3^8;

(b) the cycle structure of σ_1 is 2^{12};

(c) the cycle structure of σ_∞ is $(n_1,\dots,n_6)$;

(d) $\sigma_0 \cdot \sigma_1 \cdot \sigma_\infty$ = identity;

(e) σ_0, σ_1, and σ_∞ generate a transitive subgroup of S_{24}.

Now the proof of Theorem (X.1.1) is, unhappily, simply a list of permutations, one set for each of the 112 6-tuples. I will use the letters a,b,c,...,v,w,x as the symbols in S_{24}, and normalize the symbols so that in every case σ_0 = (abc)(def)(ghi)(jkl)(mno)(pqr)(stu)(vwx). Therefore I will only give, for a particular 6-tuple, the permutation σ_1; to verify the statement in any one case the reader must first compute $\sigma_\infty = \sigma_1^{-1} \cdot \sigma_0^{-1} = \sigma_1 \cdot$(acb)(dfe)(gih)(jlk)(mon)(prq)(sut)(vxw) and check that it has cycle structure $(n_1,\dots,n_6)$. Then one must finally check that σ_0, and σ_1

generate a transitive subgroup of S_{24} (σ_∞ is of course not needed).

The product of two permutations $\alpha \cdot \beta$ means that first β is done, then α.

(X.2.2)Table of permutations σ_1 proving the existence of the 112 6-tuples.

$[n_1 n_2 n_3 n_4 n_5 n_6]$	σ_1
[1 1 1 1 1 19]:	(ab)(cn)(df)(ek)(gi)(hl)(jm)(ow)(pq)(rx)(st)(uv)
[1 1 1 1 2 18]:	(ae)(bl)(ck)(df)(gi)(hn)(jm)(ow)(pq)(rx)(st)(uv)
[1 1 1 1 3 17]:	(al)(bh)(cn)(df)(ek)(gi)(jm)(ow)(pq)(rx)(st)(uv)
[1 1 1 1 4 16]:	(ac)(bt)(dk)(ef)(gh)(iw)(jr)(lv)(mo)(nq)(pu)(sx)
[1 1 1 1 5 15]:	(aq)(bc)(de)(fs)(gw)(hn)(iu)(jt)(kl)(mo)(pv)(rx)
[1 1 1 1 6 14]:	(an)(bk)(ch)(df)(el)(gi)(jm)(ow)(pq)(rx)(st)(uv)
[1 1 1 1 7 13]:	(as)(bc)(df)(eg)(hn)(it)(jm)(kl)(ow)(pq)(rx)(uv)
[1 1 1 1 9 11]:	(ae)(bk)(cn)(df)(gi)(hl)(jm)(ow)(pq)(rx)(st)(uv)
[1 1 1 1 10 10]:	(an)(bq)(cj)(de)(fm)(gh)(ip)(kl)(ow)(rx)(sv)(tu)
[1 1 1 2 2 17]:	(am)(bj)(cn)(df)(ek)(gq)(hl)(ir)(ow)(pu)(st)(vx)
[1 1 1 2 3 16]:	(at)(bc)(dk)(em)(fo)(gh)(iw)(jl)(nq)(px)(rs)(uv)
[1 1 1 2 4 15]:	(ab)(cq)(df)(en)(gh)(ip)(jm)(ku)(lt)(ow)(rx)(sv)
[1 1 1 2 5 14]:	(am)(bn)(cj)(df)(ek)(gq)(hl)(ir)(ow)(pu)(st)(vx)
[1 1 1 2 6 13]:	(ae)(bd)(cj)(fm)(gh)(ip)(kl)(nq)(ow)(rx)(sv)(tu)
[1 1 1 2 7 12]:	(ab)(cx)(dk)(em)(ft)(gh)(iw)(jl)(nq)(op)(rs)(uv)
[1 1 1 2 8 11]:	(ad)(bf)(cq)(en)(gh)(ip)(jm)(kl)(ow)(rx)(sv)(tu)
[1 1 1 2 9 10]:	(ak)(bn)(cj)(df)(em)(gq)(hl)(ir)(ow)(pu)(st)(vx)
[1 1 1 3 3 15]:	(ac)(bg)(de)(ft)(hm)(iq)(js)(kl)(nv)(or)(px)(uw)
[1 1 1 3 4 14]:	(am)(bt)(ci)(de)(fh)(gs)(jl)(kn)(ow)(pq)(rx)(uv)
[1 1 1 3 5 13]:	(ar)(bx)(cn)(dh)(ek)(fl)(gi)(jm)(ow)(pq)(st)(uv)
[1 1 1 3 6 12]:	(aq)(bc)(df)(eg)(hn)(ij)(kl)(mt)(ow)(ps)(rx)(uv)
[1 1 1 3 7 11]:	(av)(bn)(cf)(do)(ew)(gh)(ir)(jl)(kq)(mp)(su)(tx)
[1 1 1 3 8 10]:	(ac)(bj)(dm)(ek)(fu)(gh)(io)(lp)(nt)(qs)(rw)(vx)
[1 1 1 4 6 11]:	(ab)(co)(df)(et)(gk)(hi)(jm)(lp)(nw)(qu)(rx)(sv)
[1 1 1 4 7 10]:	(au)(bh)(cn)(df)(ek)(gi)(jm)(ls)(ow)(pq)(rx)(tv)
[1 1 1 5 5 11]:	(ap)(bj)(cn)(df)(eh)(gm)(io)(kl)(qw)(rx)(st)(uv)
[1 1 1 5 6 10]:	(ax)(br)(cn)(dh)(ek)(fl)(gi)(jm)(ow)(pq)(st)(uv)
[1 1 1 5 7 9]:	(ae)(bk)(cn)(df)(gi)(hl)(jm)(ot)(pq)(rx)(sw)(uv)
[1 1 1 6 7 8]:	(ad)(bm)(cq)(en)(fj)(gh)(ip)(kl)(ow)(rx)(sv)(tu)
[1 1 1 7 7 7]:	(ak)(br)(cn)(df)(ei)(gm)(hl)(jx)(ow)(pq)(st)(uv)
[1 1 2 2 2 16]:	(at)(bc)(dk)(em)(fo)(gh)(iw)(jx)(lv)(nq)(pu)(rs)
[1 1 2 2 3 15]:	(ad)(bg)(ce)(ft)(hi)(js)(kl)(mq)(nv)(or)(px)(uw)
[1 1 2 2 4 14]:	(ag)(bc)(df)(em)(hl)(ik)(jv)(nq)(ow)(pu)(rs)(tx)
[1 1 2 2 5 13]:	(ak)(bj)(cn)(dg)(ei)(fm)(hl)(ow)(pq)(rx)(st)(uv)
[1 1 2 2 6 12]:	(am)(bj)(cn)(ds)(ek)(ft)(gi)(hl)(ow)(pq)(rx)(uv)
[1 1 2 2 7 11]:	(ad)(bf)(cn)(ek)(gq)(hl)(ir)(jm)(ow)(pu)(st)(vx)
[1 1 2 2 9 9]:	(am)(bj)(cn)(dp)(ek)(fq)(gi)(hl)(ow)(rx)(st)(uv)
[1 1 2 3 3 14]:	(an)(bd)(cj)(em)(fk)(gq)(hl)(ir)(ow)(pu)(st)(vx)
[1 1 2 3 4 13]:	(ai)(bd)(cn)(ek)(fj)(gm)(hl)(ow)(pq)(rx)(st)(uv)
[1 1 2 3 5 12]:	(ac)(bm)(dw)(eh)(ft)(gq)(ik)(jl)(nr)(op)(sx)(uv)
[1 1 2 3 6 11]:	(ai)(bj)(cn)(df)(ek)(gm)(hl)(os)(pq)(rx)(tw)(uv)
[1 1 2 3 7 10]:	(ah)(bl)(cn)(di)(eu)(fg)(jm)(kv)(ow)(pq)(rx)(st)
[1 1 2 3 8 9]:	(ab)(cq)(df)(en)(gs)(hu)(ip)(jm)(kv)(lt)(ow)(rx)
[1 1 2 4 4 12]:	(ae)(bk)(cw)(dq)(fs)(go)(hn)(iu)(jl)(mx)(pr)(tv)
[1 1 2 4 5 11]:	(ac)(bm)(dw)(ev)(fg)(hu)(ik)(jl)(nr)(op)(qt)(sx)
[1 1 2 4 6 10]:	(ad)(bf)(cr)(ex)(gi)(hl)(jm)(kt)(ns)(ow)(pq)(uv)
[1 1 2 4 7 9]:	(ag)(bw)(ch)(dk)(ef)(it)(jq)(lv)(mo)(nr)(pu)(sx)
[1 1 2 4 8 8]:	(at)(bc)(dk)(em)(fo)(gh)(iw)(ju)(lv)(nq)(px)(rs)
[1 1 2 5 5 10]:	(aw)(bp)(cq)(de)(fr)(gi)(hs)(jv)(kx)(ln)(mu)(ot)

$[n_1n_2n_3n_4n_5n_6]$	σ_1
[1 1 2 5 6 9]:	(ab)(cq)(df)(en)(gs)(hv)(ip)(jm)(ku)(lt)(ow)(rx)
[1 1 2 5 7 8]:	(aq)(bc)(de)(fs)(gw)(hn)(iu)(jt)(km)(lo)(pv)(rx)
[1 1 2 6 6 8]:	(at)(bf)(cx)(dk)(em)(gh)(iw)(jl)(nq)(op)(rs)(uv)
[1 1 3 3 4 12]:	(af)(bj)(cn)(di)(ek)(gm)(hl)(ow)(pq)(rx)(st)(uv)
[1 1 3 3 5 11]:	(ac)(bm)(dh)(ev)(fg)(ik)(jl)(nr)(os)(px)(qt)(uw)
[1 1 3 3 8 8]:	(at)(bf)(co)(dk)(em)(gh)(iw)(jl)(nq)(px)(rs)(uv)
[1 1 3 4 4 11]:	(ao)(bh)(cj)(dg)(ew)(fs)(ip)(kl)(mv)(nq)(rx)(tu)
[1 1 3 4 6 9]:	(ah)(bl)(cf)(di)(eu)(gn)(jm)(kv)(ow)(pq)(rx)(st)
[1 1 3 4 7 8]:	(ab)(cq)(df)(es)(gn)(hu)(ip)(jm)(kv)(lt)(ow)(rx)
[1 1 3 5 6 8]:	(ai)(bj)(cn)(df)(ek)(gm)(hl)(op)(qw)(rx)(st)(uv)
[1 1 3 5 7 7]:	(ai)(bs)(cn)(df)(ek)(gm)(hl)(jt)(ow)(pq)(rx)(uv)
[1 1 4 4 7 7]:	(ao)(bh)(cj)(dg)(ew)(fm)(ip)(kl)(nq)(rx)(sv)(tu)
[1 1 4 5 6 7]:	(ac)(bg)(di)(eq)(fn)(hm)(js)(kl)(or)(px)(tv)(uw)
[1 1 4 6 6 6]:	(ac)(bh)(dm)(ek)(fq)(gv)(il)(jx)(nt)(op)(rw)(su)
[1 1 5 5 6 6]:	(ac)(bm)(dh)(ev)(fg)(ik)(jl)(nr)(op)(qt)(sx)(uw)
[1 2 2 2 3 14]:	(ak)(bc)(dj)(el)(fg)(ht)(is)(mp)(nr)(ox)(qw)(uv)
[1 2 2 2 5 12]:	(am)(bj)(cn)(ds)(ek)(ft)(gq)(hl)(ir)(ow)(pu)(vx)
[1 2 2 2 7 10]:	(ax)(bc)(dj)(el)(fg)(ht)(is)(ko)(mp)(nr)(qw)(uv)
[1 2 2 3 3 13]:	(au)(bt)(cd)(eq)(fs)(gx)(hk)(ij)(lv)(mn)(op)(rw)
[1 2 2 3 4 12]:	(ab)(cd)(eq)(fs)(gx)(hk)(ij)(lv)(mu)(nt)(op)(rw)
[1 2 2 3 5 11]:	(ak)(bv)(co)(df)(eh)(gt)(is)(jr)(lp)(mx)(nw)(qu)
[1 2 2 3 6 10]:	(at)(bj)(cl)(dk)(em)(fo)(gh)(iw)(nq)(px)(rs)(uv)
[1 2 2 3 7 9]:	(ad)(bt)(ce)(fs)(gm)(hw)(in)(ju)(kx)(lv)(op)(qr)
[1 2 2 4 5 10]:	(ac)(bl)(dk)(eh)(fg)(it)(jq)(mv)(ns)(ow)(pu)(rx)
[1 2 2 4 7 8]:	(ac)(bd)(em)(fk)(gj)(hl)(ir)(nq)(ow)(pu)(sv)(tx)
[1 2 2 5 5 9]:	(am)(bn)(cj)(ds)(ek)(ft)(gq)(hl)(ir)(ow)(pu)(vx)
[1 2 2 5 6 8]:	(ag)(bq)(cn)(df)(em)(hl)(ik)(jv)(ow)(pu)(rs)(tx)
[1 2 2 6 6 7]:	(ap)(bc)(dj)(el)(fg)(ht)(is)(ko)(mx)(nr)(qw)(uv)
[1 2 3 3 3 12]:	(ag)(bd)(cj)(em)(fk)(hl)(ir)(no)(pu)(qw)(sv)(tx)
[1 2 3 3 4 11]:	(au)(bm)(cd)(eq)(fs)(gi)(hk)(jx)(lv)(nt)(op)(rw)
[1 2 3 3 6 9]:	(ac)(bt)(dv)(ej)(fs)(gm)(hr)(in)(kx)(lu)(op)(qw)
[1 2 3 3 7 8]:	(ao)(bv)(cq)(di)(el)(fg)(hk)(jm)(np)(rx)(st)(uw)
[1 2 3 4 4 10]:	(ar)(bd)(cj)(em)(fk)(gi)(hl)(nq)(ow)(pu)(sv)(tx)
[1 2 3 4 5 9]:	(ah)(bp)(cu)(dm)(es)(fr)(gi)(jv)(kx)(ln)(ot)(qw)
[1 2 3 4 6 8]:	(ap)(bc)(dj)(ew)(fg)(ht)(is)(ko)(lq)(mx)(nr)(uv)
[1 2 3 5 6 7]:	(aj)(bv)(co)(df)(eh)(gt)(is)(kr)(lp)(mx)(nw)(qu)
[1 2 4 4 6 7]:	(ab)(cj)(dg)(em)(fk)(hl)(ir)(nq)(ow)(pu)(sv)(tx)
[1 2 4 5 5 7]:	(au)(bh)(cn)(df)(ek)(gp)(iq)(jm)(ls)(ow)(rx)(tv)
[1 2 4 5 6 6]:	(ag)(bd)(cj)(em)(fh)(ir)(kl)(nq)(ow)(pu)(sv)(tx)
[1 3 3 3 5 9]:	(ac)(bw)(dj)(ep)(fg)(ht)(ik)(lq)(mx)(nr)(os)(uv)
[1 3 3 4 5 8]:	(aq)(bd)(co)(eg)(fl)(hn)(ij)(km)(pv)(rx)(su)(tw)
[1 3 3 5 6 6]:	(ac)(bj)(dl)(ev)(fg)(hm)(ik)(nr)(os)(px)(qt)(uw)
[1 3 4 4 4 8]:	(ag)(bw)(cf)(dk)(eh)(it)(jr)(lv)(mo)(nq)(pu)(sx)
[1 3 4 4 5 7]:	(ah)(bl)(cq)(di)(eu)(fp)(gn)(jm)(kv)(ow)(rx)(st)
[2 2 2 2 8 8]:	(at)(bh)(cg)(dk)(em)(fo)(iw)(jx)(lv)(nq)(pu)(rs)
[2 2 2 3 3 12]:	(aq)(bd)(cf)(eg)(hn)(ij)(km)(lo)(ps)(rx)(tw)(uv)
[2 2 2 3 5 10]:	(ad)(bg)(ce)(ft)(hl)(ik)(js)(mq)(nv)(or)(px)(uw)
[2 2 2 4 6 8]:	(ag)(bd)(cf)(em)(hl)(ik)(jv)(nq)(ow)(pu)(rs)(tx)
[2 2 2 6 6 6]:	(am)(bj)(cn)(ds)(ek)(ft)(gp)(hl)(iq)(ow)(rx)(uv)
[2 2 3 3 4 10]:	(ae)(bd)(cj)(fk)(gm)(hl)(ir)(nq)(ow)(pu)(sv)(tx)
[2 2 3 3 7 7]:	(an)(bd)(cj)(em)(fk)(gq)(hl)(ir)(ow)(pu)(sv)(tx)
[2 2 3 4 5 8]:	(ag)(bd)(cj)(el)(fk)(hm)(ir)(nq)(ow)(pu)(sv)(tx)
[2 2 3 5 5 7]:	(ag)(bd)(cj)(em)(fo)(hl)(ir)(kw)(nq)(pu)(sv)(tx)
[2 2 4 4 4 8]:	(ag)(bt)(ch)(dk)(em)(fo)(iw)(jr)(lv)(nq)(pu)(sx)
[2 2 4 4 6 6]:	(av)(bl)(cd)(eq)(fs)(gx)(hk)(ij)(mu)(nt)(op)(rw)

$[n_1n_2n_3n_4n_5n_6]$	σ_1
[2 2 5 5 5 5]:	(an)(bl)(cs)(dk)(eh)(fg)(it)(jq)(mv)(ow)(pu)(rx)
[2 3 3 3 4 9]:	(aq)(bd)(co)(eg)(fl)(hn)(ij)(km)(ps)(rx)(tw)(uv)
[2 3 3 4 5 7]:	(ag)(bd)(cj)(em)(fk)(hl)(ir)(nq)(ow)(pu)(sv)(tx)
[2 3 3 4 6 6]:	(at)(bj)(cg)(dk)(em)(fo)(hl)(iw)(nq)(px)(rs)(uv)
[2 3 4 4 5 6]:	(ap)(bw)(ce)(dj)(fg)(ht)(is)(ko)(lq)(mx)(nr)(uv)
[3 3 3 3 6 6]:	(al)(bx)(co)(dp)(ei)(fw)(gr)(hu)(jn)(kv)(mt)(qs)
[3 3 4 4 5 5]:	(at)(bl)(cq)(di)(eu)(fp)(gn)(hs)(jm)(kv)(ow)(rx)
[4 4 4 4 4 4]:	(ao)(bt)(ch)(dk)(em)(fg)(iw)(jr)(lv)(nq)(pu)(sx)

As a remark, one can use this same method to construct the 6 Beauville surfaces, namely the 6 semistable rational elliptic surfaces with exactly 4 singular fibers of type I_n. Here one needs to find three permutations σ_0, σ_1, and σ_∞ in S_{12} with cycle structures 3^4, 2^6, and $(n_1,\ldots,n_4)$, to realize $[n_1,\ldots,n_4]$; one of course also needs $\sigma_0 \cdot \sigma_1 \cdot \sigma_\infty$ = identity and they must generate a transitive subgroup of S_{12}. 12 is a small enough number that these computations can be done by hand, and I get the following possibilities, up to conjugacy.

(X.2.3)Table of permutations demonstrating the existence of the six Beauville surfaces. σ_0 = (123)(456)(789)(abc)

$[n_i]$	σ_1	σ_∞
[9111]:	(13)(2a)(46)(5b)(79)(8c)	(1)(4)(7)(23a89c56b)
[8211]:	(13)(46)(27)(5a)(8c)(9b)	(1)(4)(9c)(237b56a8)
[6321]:	(13)(24)(57)(6a)(8c)(9b)	(1)(9c)(67b)(234a85)
[5511]:	(13)(46)(27)(8a)(9b)(5c)	(1)(4)(37b82)(6c9a5)
[4422]:	(14)(26)(37)(5a)(9b)(8c)	(24)(9c)(17b5)(36a8)
[3333]:	(14)(27)(68)(3a)(9b)(5c)	(1a5)(248)(37b)(6c9)

The reader can check that in fact these are unique up to conjugacy, and so gives a proof of the uniqueness of the 6 Beauville surfaces.

In the K3 case, it is not so easy to do the computations by hand: they are due to U. Persson and I, and we used a computer to generate them. We do not know that they are unique up to conjugacy: this is an open problem.

Let us turn to proving that the rest of the s-tuples do not exist, in the K3 case.

X.3: Applications of the existence of torsion

Assume that $\pi:X \longrightarrow \mathbb{P}^1$ is a semistable K3 elliptic surface with section S_0 realizing an s-tuple $[n_1,\ldots,n_s]$. For each I_n singular fiber of π, label the components $C_0,\ldots,C_{n-1}$ going around the cycle, with C_0 being the component meeting S_0. Using the notation of Lecture VII, we have that $G_R \cong \mathbb{Z}/n_1\times\ldots\times\mathbb{Z}/n_s$ (Lemma (VII.2.6)); moreover in this identification the coset of the dual element to C_k is represented by k (mod n_i) in the i^{th} factor. Any torsion section S induces a class in G_R, and by Corollary (VII.3.1), this assignment is 1-1; hence a section is determined by an s-tuple ($k_1$mod $n_1,\ldots,k_s$mod n_s), where this means that S meets C_{k_j} in the j^{th} singular fiber. Not all such s-tuples are achieved, however, and it is a tricky problem to determine the torsion subgroup TMW of the Mordell-Weil group of sections; in any case we can view TMW as a subgroup H of G_R, as described in section 3 of Lecture IX. Since this subgroup must be totally isotropic for the quadratic form q on G_R described in section 1 of Lecture IX, we must have that

(X.3.1) if $S \in$ TMW represents $(k_1,\ldots,k_s) \in G_R$, then $\sum\frac{(n_i-1)}{2n_i} k_i^2 \in \mathbb{Z}$;

this follows by the computation (IX.1.4).

Assume S is a torsion section of prime order p. Let us denote by τ_S the automorphism of X given by translation by S; τ_S has order p. Let Y denote the minimal resolution of singularities of the quotient X/τ_S. Since τ_S preserves the fibers of π, and on a smooth fiber, τ_S restricts to translation by a torsion point of order p (which has as quotient a smooth elliptic curve), we see that π induces an elliptic fibration $\pi_S:X/\tau_S \longrightarrow \mathbb{P}^1$; π_S has at most s singular fibers, underneath the s singular fibers of π.

Our first task is to determine the singular fibers of the desingularization Y. The question is local on the base curve, so let us focus on the i^{th} singular fiber I_{n_i} of π. Assume S represents $(k_1,\ldots,k_s)$, so that S meets C_{k_i} in the i^{th}fiber.

If k_i is not zero, then τ_S, when restricted to this singular fiber, is a translation by a point which is not in the connected component C_0 of the identity in the group law on the smooth points of the fiber. Therefore τ_S must permute the components, and since the group of components is isomorphic to $\mathbb{Z}/n_i$, (Lemma (VII.3.5)), we see that τ_S simply "rotates" the fiber components, by the amount k_i: the points of C_j are sent to the points of C_{j+k_i}. In particular, there are no fixed points to τ_S on this fiber, and the quotient X/τ_S is smooth in a neighborhood of this fiber, with a singular fiber of type $I_{n_i/p}$. (Note that if S has order p, then $k_i = rn_i/p$ for some r, and the orbits of translation by S have size p.) Note finally that if k_i is not zero, then n_i must be divisible by p.

If k_i is zero, then τ_S is translation by a point of order p in C_0, and hence preserves the components C_j; in each component (whose smooth points form a torsor under the smooth points of C_0, which is a group isomorphic to $\mathbb{C}^*$), translation by τ_S acts as multiplication by a nontrivial p^{th} root of unity. In particular, the n_i nodes of the fiber are fixed under τ_S, and the fiber of X/τ_S under this fiber is a cycle of n_i $\mathbb{P}^1$'s, meeting in a cycle. However since the local action of τ_S at a node can be given by a 2×2 matrix $\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}$ for some p^{th} root of unity ζ, the surface X/τ_S has A_{p-1} singularities at the n_i nodes of the cycle. Therefore the desingularization Y "inserts" a chain of p-1 $\mathbb{P}^1$'s at every node, giving a total of $n_i + (p-1)(n_i) = n_i p$ $\mathbb{P}^1$'s in a cycle. Therefore Y has a singular fiber of type $I_{n_i p}$.

Summarizing, we have the following.

(X.3.2)<u>Lemma:</u> Let S be a torsion section of π of prime order p, representing $(k_1,\ldots,k_s)$ in G_R. Let Y be the desingularization of X/τ_S. Then π induces an elliptic fibration on Y, which also has exactly s singular fibers of type I_n, one underneath each singular fiber of X. Moreover:

(a) If $k_i \neq 0$ then $p|n_i$ and the fiber of Y is of type $I_{n_i/p}$.

(b) If $k_i = 0$ then the fiber of Y is of type $I_{n_i p}$.

(c) Y is again a K3 surface.

Proof: Only (c) needs any further remarks. We only need to prove that the holomorphic 2-form on X descends to X/τ_S, since X/τ_S has only rational singularities; but this follows since there are only a finite number of fixed points for the action. ■

(X.3.3)Corollary: Let S be a torsion section of π of prime order p, representing $(k_1,\ldots,k_s)$ in G_R. Then $\sum\{n_i|k_i = 0\} = 24/(p+1)$.

Proof: Since Y is again a K3 surface, the sum of the subscripts on its singular fibers must be 24; hence $\sum\{n_i/p|k_i\neq 0\}+\sum\{n_i p|k_i=0\} = 24$. Multiplying through by p and noting that $\sum n_i = 24$ gives $24 + (p^2-1)\sum\{n_i|k_i=0\} = 24p$, which proves the result. ∎

This leads to the following useful condition for the existence of a torsion section of prime order.

(X.3.4)Corollary:(The Fixed Point Rule): Let X be a semistable elliptic K3 surface realizing $[n_1,\ldots,n_s]$, and assume that X has a torsion section of prime order p. Then some subset of $\{n_i|p|n_i\}$ must sum to $24p/(p+1)$.

Proof: From the previous Corollary,
$\sum\{n_i|k_i \neq 0\} = 24 - 24/(p+1) = 24p/(p+1)$, and so the Fixed Point Rule follows by recalling that if $k_i \neq 0$ then $p|n_i$. ∎

The Fixed Point Rule is the main tool for determining that a K3 elliptic surface X realizing $[n_i]$ cannot have a torsion section of some particular prime order. This, combined with the Length and Discriminant Criteria (Propositions (IX.3.1) and (IX.3.2)), which force torsion sections to exist, form the backbone of the proofs on non-existence of the s-tuples $[n_i]$ listed in section 1.

X.4: The 135 impossible s-tuples

In this section (and those following) I will present the proofs of the non-existence of the s-tuples as described in section one. We work "from the bottom up", and begin with the three 9-tuples which are not yet proved to exist: in fact they do not.

(X.4.1)Proposition: The three 9-tuples
[1,2,2,2,2,2,2,2,9], [2,2,2,2,2,2,2,3,7], and [2,2,2,2,2,2,2,5,5]
do not exist.

Proof: By the Length Criterion (Proposition (IX.3.1)), if any of these existed, there would be 2-torsion in TMW, i.e., a section of order 2 would exist. However in each of these cases the Fixed Point Rule is violated: for p = 2, 24p/(p+1) = 16, and the sum of all the even n_i's is only 14 in each case. ■

Once we know that one s-tuple does not exist, we can apply Lemma (VIII.2.1) to conclude that any "degeneration" of that s-tuple (i.e., one obtained by replacing two of the n_i's with their sum) can not exist. We will repeatedly use this remark, and "bootstrap" our way into the next lower s-level.

Our first application is to eliminate 7 of the 8-tuples, namely [1,2,2,2,2,2,2,11], [1,2,2,2,2,2,4,9], [2,2,2,2,2,2,2,10], [2,2,2,2,2,2,3,9], [2,2,2,2,2,2,5,7], [2,2,2,2,2,3,4,7], and [2,2,2,2,2,4,5,5]; each of these is an obvious degeneration of one of the three impossible 9-tuples. This leaves four remaining 8-tuples to discuss, namely [1,1,1,4,4,4,4,5], [1,1,3,3,3,3,3,7], [1,3,3,3,3,3,4,4], and [2,2,3,3,3,3,3,5].

For the latter three, the Length Criterion implies that there must be 3-torsion in TMW. For p = 3, 24p/(p+1) = 18, so some subset of the numbers divisible by 3 must sum to 18 in these cases. However the total sum of all numbers divisible by 3 is 15 in each of these cases: this contradiction shows that these are impossible.

This leaves only the last case [1,1,1,4,4,4,4,5]. Here the Discriminant Criterion (Proposition (IX.3.2)) implies that there must be 2-torsion in TMW. Let S be a section of order 2. It must represent the element (0,0,0,2,2,2,2,0) in G_R, by Lemma (X.3.2(a)) and Corollary (X.3.3). The quotient by τ_S gives a surface Y realizing the 8-tuple [2,2,2,2,2,2,2,10], by Lemma (X.3.2); this we have seen above does not exist, since it deforms to [1,2,2,2,2,2,2,2,9]. This contradiction proves that [1,1,1,4,4,4,4,5] does not exist, and finishes the analysis of the 8-tuples:

(X.4.2)<u>Proposition:</u> The 11 8-tuples listed in Corollary (X.1.3) do not exist.

Let us turn our attention to the 7-tuples. Of the 34 7-tuples listed in Corollary (X.1.2), 31 of them are degenerations of one of the 11 impossible 8-tuples. Therefore these 31 do not exist, and we are left to analyze the remaining three 7-tuples [1,1,3,4,5,5,5], [1,2,3,3,5,5,5], and

[2,3,3,4,4,4,4].

The first two must have 5-torsion by the Discriminant Criterion. For $p = 5$, $24p/(p+1) = 20$; since the sum of the numbers in both of these 7-tuples divisible by 5 add up to only 15, we obtain a contradiction using the Fixed Point Rule. Therefore these two do not exist.

We are left then to discuss [2,3,3,4,4,4,4]. By the Fixed Point Rule, the only possible torsion is 2-torsion, so TMW is a finite abelian 2-group. The Length Criterion forces at least one section S of order two, and the only possibility is that S represents the element (0,0,0,2,2,2,2) in G_R, according to Lemma (X.3.2(a)) and Corollary (X.3.3). Note that since there is then a unique element of order two, TMW is a cyclic 2-group.

Assume that TMW has an element S_1 of order 4. Then $2S_1$ must represent (0,0,0,2,2,2,2) in G_R, so S_1 must represent an element of the form (0 or $1,0,0,\pm1,\pm1,\pm1,\pm1$) in G_R. However none of these elements are isotropic: they all violate (X.3.1). Hence TMW has no element of order four, so since TMW is a cyclic 2-group, we have that TMW has order 2, consisting of the zero section S_0 and the order two section S obtained above.

Let H be the isotropic subgroup of G_R corresponding to TMW; $H = \{\underline{0}, (0,0,0,2,2,2,2)\}$. A calculation shows that $H^{\perp}/H$ is abstractly isomorphic to $(\mathbb{Z}/2)^3 \times (\mathbb{Z}/4)^2 \times (\mathbb{Z}/3)^2$, so that the length of $H^{\perp}/H$ is 5.

We now obtain a contradiction via the following general fact.

(X.4.3)<u>Lemma:</u> Let $\pi : X \longrightarrow \mathbb{P}^1$ be a semistable elliptic K3 surface realizing the s-tuple $[n_1, \ldots, n_s]$. Assume that the group of torsion sections TMW corresponds to the isotropic subgroup H of G_R. Then $\mathrm{length}(H^{\perp}/H) \le s - 4$.

Proof: Let U be the unimodular rank 2 sublattice of $H^2(X,\mathbb{Z})$ generated by the zero section S_0 and the fiber F. The lattice R generated by components of fibers not meeting S_0 sits inside $U^{\perp}$, with quotient equal to the Mordell-Weil group MW(X), by (VII.2.5); moreover, the torsion part TMW(X) is naturally identified with $H = R^{\perp\perp}/R$. Since $H^2(X,\mathbb{Z})$ and U are both unimodular, so is $U^{\perp}$, and since $H^2(X,\mathbb{Z})$ has rank 22, $U^{\perp}$ has rank 20. Note that R has rank $\sum(n_i - 1) = 24 - s$, so that $R^{\perp}$ (taken inside $U^{\perp}$) has rank $20 - (24 - s) = s - 4$.

Therefore

$$\begin{aligned} \text{length}(H^{\perp}/H) &= \text{length}(G_{R^{\perp\perp}}) && \text{(by Lemma (IX.1.6))} \\ &= \text{length}(G_{R^{\perp}}) && \text{(by Lemma (IX.1.7))} \\ &\le \text{rank}(R^{\perp}) && \text{(by (IX.1.2))} \\ &= s - 4. \quad \blacksquare \end{aligned}$$

In the case under discussion, we have $s = 7$ and $\text{length}(H^{\perp}/H) = 5$, violating the lemma. Hence [2,3,3,4,4,4,4] does not exist. Thus:

(X.4.4)<u>Proposition:</u> The 34 7-tuples listed in Corollary (X.1.2) do not exist.

Finally we deal with the 87 6-tuples not yet known to exist, namely the complement of the 112 6-tuples listed in Theorem (X.1.1). Of these 87 6-tuples, 78 of them are degenerations of one of the 34 7-tuples now known not to exist; therefore these 78 do not exist, and we are left to discuss the following 9 separately: [1,1,1,1,8,12], [1,1,1,3,9,9], [1,1,1,6,6,9], [1,1,2,2,8,10], [1,1,2,6,7,7], [1,2,2,3,8,8], [1,2,2,5,7,7], [1,2,3,4,7,7], and [1,2,3,6,6,6].

For [1,1,1,1,8,12], the Discriminant Criterion forces a 2-torsion section, violating the Fixed Point Rule. For both [1,1,1,3,9,9] and [1,1,1,6,6,9], the Length Criterion forces a 3-torsion section. In the case of [1,1,1,3,9,9], a section of order 3 must be $(0,0,0,0,\pm3,\pm3)$ according to Lemma (X.3.2) and Corollary (X.3.3); the quotient by such a section would then realize [3,3,3,9,3,3] by Lemma (X.3.2), and this is one of the 78 already eliminated: indeed, it obviously deforms to [1,1,3,3,3,3,3,7], which is an 8-tuple previously eliminated. In the case of [1,1,1,6,6,9], the existence of 3-torsion violates the Fixed Point Rule.

For both [1,1,2,2,8,10] and [1,2,3,6,6,6], the Length Criterion forces a 2-torsion section, violating the Fixed Point Rule. For each of [1,1,2,6,7,7], [1,2,2,5,7,7], and [1,2,3,4,7,7], the Discriminant Criterion forces a 7-torsion section, violating the Fixed Point Rule.

This leaves only [1,2,2,3,8,8], which by the Length Criterion must have a section of order 2, which must be (0,0,0,0,4,4) by Lemma (X.3.2) and Corollary (X.3.3). The quotient then realizes [2,4,4,6,4,4], which is again one of the 78 previously eliminated: it is a degeneration of [2,3,3,4,4,4,4], which is a 7-tuple eliminated above.

This completes the analysis of the 6-tuples:

(X.4.5) <u>Proposition:</u> The only 6-tuples which exist are those listed in Theorem (X.1.1).

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I have not been overly generous in my use of references in the text; I found that except for some basic facts about surfaces and fibrations, I wanted to make the notes self-contained enough so that the reader could follow the discussion without frequent infusions of foreign material. However, the time has come to remedy this situation, and I feel compelled to indicate not only alternate viewpoints on this basic material, but also some glimpse of the extensive advanced literature on elliptic surfaces.

I do not mean to imply that the list below is in any sense complete, but it does certainly contain the basic references.

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